

KAC-MOODY LIE ALGEBRAS GRADED BY KAC-MOODY ROOT SYSTEMS

HECHMI BEN MESSAOUD, GUY ROUSSEAU

ABSTRACT. We look to gradations of Kac-Moody Lie algebras by Kac-Moody root systems with finite dimensional weight spaces. We extend, to general Kac-Moody Lie algebras, the notion of C -admissible pair as introduced by H. Rubenthaler and J. Nervi for semi-simple and affine Lie algebras. If \mathfrak{g} is a Kac-Moody Lie algebra (with Dynkin diagram indexed by I) and (I, J) is such a C -admissible pair, we construct a C -admissible subalgebra \mathfrak{g}^J , which is a Kac-Moody Lie algebra of the same type as \mathfrak{g} , and whose root system Σ grades finitely the Lie algebra \mathfrak{g} . For an admissible quotient $\rho : I \rightarrow \overline{I}$ we build also a Kac-Moody subalgebra \mathfrak{g}^ρ which grades finitely the Lie algebra \mathfrak{g} . If \mathfrak{g} is affine or hyperbolic, we prove that the classification of the gradations of \mathfrak{g} is equivalent to those of the C -admissible pairs and of the admissible quotients. For general Kac-Moody Lie algebras of indefinite type, the situation may be more complicated; it is (less precisely) described by the concept of generalized C -admissible pairs.

2000 Mathematics Subject Classification. 17B67.

Key words and phrases. Kac-Moody algebra, C -admissible pair, gradation.

Introduction. The notion of gradation of a Lie algebra \mathfrak{g} by a finite root system Σ was introduced by S. Berman and R. Moody [7] and further studied by G. Benkart and E. Zelmanov [5], E. Neher [14], B. Allison, G. Benkart and Y. Gao [1] and J. Nervi [15]. This notion was extended by J. Nervi [16] to the case where \mathfrak{g} is an affine Kac-Moody algebra and Σ the (infinite) root system of an affine Kac-Moody algebra; in her two articles she uses the notion of C -admissible subalgebra associated to a C -admissible pair for the Dynkin diagram, as introduced by H. Rubenthaler [19].

We consider here a general Kac-Moody algebra \mathfrak{g} (indecomposable and symmetrizable) and the root system Σ of a Kac-Moody algebra. We say that \mathfrak{g} is finitely Σ -graded if \mathfrak{g} contains a Kac-Moody subalgebra \mathfrak{m} (the grading subalgebra) whose root system relatively to a Cartan subalgebra \mathfrak{a} of \mathfrak{m} is Σ and moreover the action of $ad(\mathfrak{a})$ on \mathfrak{g} is diagonalizable with weights in $\Sigma \cup \{0\}$ and finite dimensional weight spaces, see Definition 1.4. The finite dimensionality of weight spaces is a new condition, it was fulfilled by the non trivial examples of J. Nervi [16] but it excludes the gradings of infinite dimensional Kac-Moody algebras by finite root systems as in [5]. Many examples of these gradations are provided by the almost split real forms of \mathfrak{g} , cf. 1.7. We are interested in describing the possible gradations of a given Kac-Moody algebra (as in [15], [16]), not in determining all the Lie algebras graded by a given root system Σ (as e.g. in [1] for Σ finite).

Let I be the index set of the Dynkin diagram of \mathfrak{g} , we generalize the notion of C -admissible pair (I, J) as introduced by H. Rubenthaler [19] and J. Nervi [15], [16], cf. Definition 2.1. For each Dynkin diagram I the classification of the C -admissible pairs (I, J) is easy to deduce from the list of irreducible C -admissible pairs due to these authors. We are able then to generalize in section 2 their construction of a C -admissible subalgebra (associated to a C -admissible pair) which grades finitely \mathfrak{g} :

Theorem 1. (cf. 2.6, 2.11, 2.14) *Let \mathfrak{g} be an indecomposable and symmetrizable Kac-Moody algebra, associated to a generalized Cartan matrix $A = (a_{i,j})_{i,j \in I}$. Let $J \subset I$ be a subset of finite type such that the pair (I, J) is C -admissible. There is a generalized Cartan matrix $A^J = (a'_{k,l})_{k,l \in I'}$ with index set $I' = I \setminus J$ and a Kac-Moody subalgebra \mathfrak{g}^J of \mathfrak{g} associated to A^J , with root system Δ^J . Then \mathfrak{g} is finitely Δ^J -graded with grading subalgebra \mathfrak{g}^J .*

For a general finite gradation of \mathfrak{g} with grading subalgebra \mathfrak{m} , we prove (in section 3) that \mathfrak{m} is also symmetrizable and the restriction to \mathfrak{m} of the invariant bilinear form of \mathfrak{g} is nondegenerate (Corollary 3.16). The Kac-Moody algebras \mathfrak{g} and \mathfrak{m} have the same type: finite, affine or indefinite; the first two types correspond to the cases already studied e.g. by J. Nervi. Moreover if \mathfrak{g} is indefinite Lorentzian or hyperbolic, then so is \mathfrak{m} (propositions 3.3 and 3.21). We get also the following precise structure result for this general situation:

Theorem 2. *Let \mathfrak{g} be an indecomposable and symmetrizable Kac-Moody algebra, finitely graded by a root system Σ of Kac-Moody type with grading subalgebra \mathfrak{m} .*

- 1) *We may choose the Cartan subalgebras \mathfrak{a} of \mathfrak{m} , \mathfrak{h} of \mathfrak{g} such that $\mathfrak{a} \subset \mathfrak{h}$. Then there is a surjective map $\rho_a : \Delta \cup \{0\} \rightarrow \Sigma \cup \{0\}$ between the corresponding root systems. We may choose the bases $\Pi_a = \{\gamma_s \mid s \in \bar{I}\} \subset \Sigma$ and $\Pi = \{\alpha_i \mid i \in I\} \subset \Delta$ of these root systems such that $\rho_a(\Delta^+) \subset \Sigma^+ \cup \{0\}$ and $\{\alpha \in \Delta \mid \rho_a(\alpha) = 0\} = \Delta_J := \Delta \cap (\sum_{j \in J} \mathbb{Z}\alpha_j)$ for some subset $J \subset I$ of finite type.*
- 2) *Let $I'_{re} = \{i \in I \mid \rho_a(\alpha_i) \in \Pi_a\}$, $I'_{im} = \{i \in I \mid \rho_a(\alpha_i) \notin \Pi_a \cup \{0\}\}$. Then $J = \{i \in I \mid \rho_a(\alpha_i) = 0\}$. We note I_{re} (resp. J°) the union of the connected components of $I \setminus I'_{im} = I'_{re} \cup J$ meeting I'_{re} (resp. contained in J), and $J_{re} = J \cap I_{re}$. Then the pair (I_{re}, J_{re}) is C -admissible.*
- 3) *There is a Kac-Moody subalgebra $\mathfrak{g}(I_{re})$ of \mathfrak{g} , associated to I_{re} , which contains \mathfrak{m} . This Lie algebra is finitely $\Delta(I_{re})^{J_{re}}$ -graded, with grading subalgebra $\mathfrak{g}(I_{re})^{J_{re}}$. Both algebras $\mathfrak{g}(I_{re})$ and $\mathfrak{g}(I_{re})^{J_{re}}$ are finitely Σ -graded with grading subalgebra \mathfrak{m} .*

It may happen that I'_{im} is non empty, we then say that (I, J) is a generalized C -admissible pair. We give and explain precisely an example in section 5.

When I'_{im} is empty, $I_{re} = I$, $J_{re} = J$, $\mathfrak{g}(I_{re}) = \mathfrak{g}$, $(I, J) = (I_{re}, J_{re})$ is a C -admissible pair and the situation looks much like the one described by J. Nervi in the finite [15] or affine [16] cases. Actually we prove that this is always true when \mathfrak{g} is of finite type, affine or hyperbolic (Proposition 3.20). In this "empty" case we get the gradation of \mathfrak{g} with two levels: \mathfrak{g} is finitely Δ^J -graded with grading subalgebra \mathfrak{g}^J as in Theorem 1 and \mathfrak{g}^J is finitely Σ -graded with grading subalgebra \mathfrak{m} . But the gradation of \mathfrak{g}^J by Σ and \mathfrak{m} is such that the corresponding set " J " described as in Theorem 2 is empty; we say (following [15], [16]) that it is a maximal gradation, cf. Definition 3.15 and Proposition 3.22.

To get a complete description of the case I'_{im} empty, it remains to describe the maximal gradations; this is done in section 4. We prove in Proposition 4.1 that a maximal gradation $(\mathfrak{g}, \Sigma, \mathfrak{m})$ is entirely described by a quotient map $\rho : I \rightarrow \bar{I}$ which is admissible i.e. satisfies two simple conditions (MG1) and (MG2) with respect to the generalized Cartan matrix $A = (a_{i,j})_{i,j \in I}$. Conversely for any admissible quotient map ρ , it is possible to build a maximal gradation of \mathfrak{g} associated to this map, cf. Proposition 4.5 and Remark 4.7.

1. PRELIMINARIES

We recall the basic results on the structure of Kac-Moody Lie algebras and we set the notations. More details can be found in the book of Kac [11]. We end by the definition of finitely graded Kac-Moody algebras.

1.1. Generalized Cartan matrices. Let I be a finite index set. A matrix $A = (a_{i,j})_{i,j \in I}$ is called a *generalized Cartan matrix* if it satisfies :

- (1) $a_{i,i} = 2 \quad (i \in I)$
- (2) $a_{i,j} \in \mathbb{Z}^- \quad (i \neq j)$
- (3) $a_{i,j} = 0$ implies $a_{j,i} = 0$.

The matrix A is called *decomposable* if for a suitable permutation of I it takes the form $\begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$ where B and C are square matrices. If A is not decomposable, it is called *indecomposable*.

The matrix A is called *symmetrizable* if there exists an invertible diagonal matrix $D = \text{diag}(d_i, i \in I)$ such that DA is symmetric. The entries $d_i, i \in I$, can be chosen to be positive rational and if moreover the matrix A is indecomposable, then these entries are unique up to a constant factor.

Any indecomposable generalized Cartan matrices is of one of the three mutually exclusive types : *finite*, *affine* and *indefinite* ([11], chap. 4).

An indecomposable and symmetrizable generalized Cartan matrix A is called *Lorentzian* if it is non-singular and the corresponding symmetric matrix has signature $(+ + \dots + -)$.

An indecomposable generalized Cartan matrix A is called *strictly hyperbolic* (resp. *hyperbolic*) if the deletion of any one vertex, and the edges connected to it, of the corresponding Dynkin diagram yields a disjoint union of Dynkin diagrams of finite (resp. finite or affine) type.

Note that a symmetrizable hyperbolic generalized Cartan matrix is non singular and Lorentzian (cf. [13]).

1.2. Kac-Moody algebras and groups. (See [11] and [17]).

Let $A = (a_{i,j})_{i,j \in I}$ be an indecomposable and symmetrizable generalized Cartan matrix. Let $(\mathfrak{h}_{\mathbb{R}}, \Pi = \{\alpha_i, i \in I\}, \Pi^* = \{\check{\alpha}_i, i \in I\})$ be a realization of A over the real field \mathbb{R} : thus $\mathfrak{h}_{\mathbb{R}}$ is a real vector space such that $\dim \mathfrak{h}_{\mathbb{R}} = |I| + \text{corank}(A)$, Π and Π^* are linearly independent in $\mathfrak{h}_{\mathbb{R}}^*$ and $\mathfrak{h}_{\mathbb{R}}$ respectively such that $\langle \alpha_j, \check{\alpha}_i \rangle = a_{i,j}$. Let $\mathfrak{h} = \mathfrak{h}_{\mathbb{R}} \otimes \mathbb{C}$, then $(\mathfrak{h}, \Pi, \Pi^*)$ is a realization of A over the complex field \mathbb{C} .

It follows that, if A is non-singular, then Π^* (resp. Π) is a basis of \mathfrak{h} (resp. \mathfrak{h}^*); moreover $\mathfrak{h}_{\mathbb{R}} = \{h \in \mathfrak{h} \mid \alpha_i(h) \in \mathbb{R}, \forall i \in I\}$ is well defined by the realization $(\mathfrak{h}, \Pi, \Pi^*)$.

Let $\mathfrak{g} = \mathfrak{g}(A)$ be the complex Kac-Moody algebra associated to A : it is generated by $\{\mathfrak{h}, e_i, f_i, i \in I\}$ with the following relations

$$(1.1) \quad \begin{aligned} [\mathfrak{h}, \mathfrak{h}] &= 0, & [e_i, f_j] &= \delta_{i,j} \alpha_i & (i, j \in I); \\ [h, e_i] &= \langle \alpha_i, h \rangle e_i, & [h, f_i] &= -\langle \alpha_i, h \rangle f_i & (h \in \mathfrak{h}); \\ (\text{ade}_i)^{1-a_{i,j}}(e_j) &= 0, & (\text{ad} f_i)^{1-a_{i,j}}(f_j) &= 0 & (i \neq j). \end{aligned}$$

The Kac-Moody algebra $\mathfrak{g} = \mathfrak{g}(A)$ is called of finite, affine or indefinite type if the corresponding generalized Cartan matrix A is.

The derived algebra \mathfrak{g}' of \mathfrak{g} is generated by the *Chevalley generators* $e_i, f_i, i \in I$, and the center \mathfrak{c} of \mathfrak{g} lies in $\mathfrak{h}' = \mathfrak{h} \cap \mathfrak{g}' = \sum_{i \in I} \mathbb{C} \alpha_i$. If the generalized Cartan matrix A is non-singular, then $\mathfrak{g} = \mathfrak{g}'$ is a (finite or infinite)-dimensional simple Lie algebra, and the center \mathfrak{c} is trivial.

The subalgebra \mathfrak{h} is a maximal $\text{ad}(\mathfrak{g})$ -diagonalizable subalgebra of \mathfrak{g} , it is called the *standard Cartan subalgebra* of \mathfrak{g} . Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ be the corresponding root system; then Π is a root basis of Δ and $\Delta = \Delta^+ \cup \Delta^-$, where $\Delta^\pm = \Delta \cap \mathbb{Z}^\pm \Pi$ is the set of positive (or negative) roots relative to the basis Π . For $\alpha \in \Delta$, let \mathfrak{g}_α be the root space of \mathfrak{g} corresponding to the root α ; then $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha)$.

The *Weyl group* W of $(\mathfrak{g}, \mathfrak{h})$ is generated by the fundamental reflections r_i ($i \in I$) such that $r_i(h) = h - \langle \alpha_i, h \rangle \alpha_i$ for $h \in \mathfrak{h}$, it is a Coxeter group on $\{r_i, i \in I\}$ with length function $w \mapsto l(w)$, $w \in W$. The Weyl group W acts on \mathfrak{h}^* and Δ , we set $\Delta^{re} = W(\Pi)$ (the real roots) and $\Delta^{im} = \Delta \setminus \Delta^{re}$ (the imaginary roots). Any root basis of Δ is W -conjugate to Π or $-\Pi$.

A *Borel subalgebra* of \mathfrak{g} is a maximal completely solvable subalgebra. A *parabolic subalgebra* of \mathfrak{g} is a (proper) subalgebra containing a Borel subalgebra. The *standard positive (or negative) Borel subalgebra* is $\mathfrak{b}^\pm := \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta^\pm} \mathfrak{g}_\alpha)$. A parabolic subalgebra \mathfrak{p}^+ (resp. \mathfrak{p}^-) containing \mathfrak{b}^+ (resp. \mathfrak{b}^-) is called *positive (resp. negative) standard parabolic subalgebra* of \mathfrak{g} ; then there exists a subset J of I (called the type of \mathfrak{p}^\pm) such that $\mathfrak{p}^\pm = \mathfrak{p}^\pm(J) := (\bigoplus_{\alpha \in \Delta_J} \mathfrak{g}_\alpha) + \mathfrak{b}^\pm$, where $\Delta_J = \Delta \cap (\bigoplus_{j \in J} \mathbb{Z} \alpha_j)$ (cf. [12]).

In [17], D.H. Peterson and V.G. Kac construct a group G , which is the connected and simply connected complex algebraic group associated to \mathfrak{g} when \mathfrak{g} is of finite type, depending only on the derived Lie algebra \mathfrak{g}' and acting on \mathfrak{g} via the adjoint representation $\text{Ad} : G \rightarrow \text{Aut}(\mathfrak{g})$. It is generated by the one-parameter subgroups $U_\alpha = \exp(\mathfrak{g}_\alpha)$, $\alpha \in \Delta^{re}$, and $\text{Ad}(U_\alpha) = \exp(\text{ad} \mathfrak{g}_\alpha)$. In the definitions of J. Tits [20] G is the group of complex points of \mathfrak{G}_D where D is the datum associated to A and the \mathbb{Z} -dual Λ of $\bigoplus_{i \in I} \mathbb{Z} \alpha_i^\vee$.

The Cartan subalgebras of \mathfrak{g} are G -conjugate. If \mathfrak{g} is not of finite type, there are exactly two conjugate classes (under the adjoint action of G) of Borel subalgebras : $G \cdot \mathfrak{b}^+$ and $G \cdot \mathfrak{b}^-$. A Borel subalgebra \mathfrak{b} of \mathfrak{g} which is G -conjugate to \mathfrak{b}^+ (resp. \mathfrak{b}^-) is called *positive (resp. negative)*. It follows that any parabolic subalgebra \mathfrak{p} of \mathfrak{g} is G -conjugate to a standard positive (or negative) parabolic subalgebra, in which case, we say that \mathfrak{p} is positive (or negative).

1.3. Standard Kac-Moody subalgebras and subgroups. Let J be a nonempty subset of I . Consider the generalized Cartan matrix $A_J = (a_{i,j})_{i,j \in J}$.

Definition 1.1. The subset J is called of finite, affine or indefinite type if the corresponding generalized Cartan matrix A_J is. We say also that J is connected, if the Dynkin subdiagram, with vertices indexed by J , is connected, or equivalently, the corresponding generalized Cartan submatrix A_J is indecomposable.

Proposition 1.2. *Let $\Pi_J = \{\alpha_j, j \in J\}$ and $\Pi_{\check{J}} = \{\alpha_{\check{j}}, j \in J\}$. Let \mathfrak{h}'_J be the subspace of \mathfrak{h} generated by $\Pi_{\check{J}}$, and $\mathfrak{h}^J = \Pi_J^\perp = \{h \in \mathfrak{h}, \langle \alpha_j, h \rangle = 0, \forall j \in J\}$. Let \mathfrak{h}''_J be a supplementary subspace of $\mathfrak{h}'_J + \mathfrak{h}^J$ in \mathfrak{h} and let*

$$\mathfrak{h}_J = \mathfrak{h}'_J \oplus \mathfrak{h}''_J,$$

then, we have :

- 1) $(\mathfrak{h}_J, \Pi_J, \Pi_{\check{J}})$ is a realization of the generalized Cartan matrix A_J . Hence $\mathfrak{h}''_J = \{0\}$, $\mathfrak{h}_J = \mathfrak{h}'_J$ when A_J is regular (e.g. when J is of finite type).
- 2) The subalgebra $\mathfrak{g}(J)$ of \mathfrak{g} , generated by \mathfrak{h}_J and the $e_j, f_j, j \in J$, is the Kac-Moody Lie algebra associated to the realization $(\mathfrak{h}_J, \Pi_J, \Pi_{\check{J}})$ of A_J .
- 3) The corresponding root system $\Delta(J) = \Delta(\mathfrak{g}(J), \mathfrak{h}_J)$ can be identified with $\Delta_J := \Delta \cap (\oplus_{j \in J} \mathbb{Z}\alpha_j)$.

N.B. The derived algebra $\mathfrak{g}'(J)$ of $\mathfrak{g}(J)$ is generated by the e_j, f_j for $j \in J$; it does not depend of the choice of \mathfrak{h}''_J .

Proof.

- 1) Note that $\dim(\mathfrak{h}''_J) = \dim(\mathfrak{h}'_J \cap \mathfrak{h}^J) = \text{corank}(A_J)$. In particular, $\dim(\mathfrak{h}_J) - |J| = \text{corank}(A_J)$. If $\alpha \in \text{Vect}(\alpha_j, j \in J)$, then α is entirely determined by its restriction to \mathfrak{h}_J and hence Π_J defines, by restriction, a free family in \mathfrak{h}_J^* . As $\Pi_{\check{J}}$ is free, assertion 1) holds.

Assertions 2) and 3) are straightforward. \square

In the same way, the subgroup G_J of G generated by $U_{\pm\alpha_j}, j \in J$, is equal to the Kac-Moody group associated to the generalized Cartan matrix A_J : it is clearly a quotient; the equality is proven in [18, 5.15.2].

1.4. The invariant bilinear form. (See [11]).

We recall that the generalized Cartan matrix A is supposed indecomposable and symmetrizable. There exists a nondegenerate $\text{ad}(\mathfrak{g})$ -invariant symmetric \mathbb{C} -bilinear form (\cdot, \cdot) on \mathfrak{g} , which is entirely determined by its restriction to \mathfrak{h} , such that

$$(\alpha_{\check{i}}, h) = \frac{(\alpha_{\check{i}}, \alpha_{\check{i}})}{2} \langle \alpha_i, h \rangle, \quad i \in I, h \in \mathfrak{h},$$

and we may thus assume that

$$(1.2) \quad (\alpha_{\check{i}}, \alpha_{\check{i}}) \text{ is a positive rational for all } i.$$

The nondegenerate invariant bilinear form (\cdot, \cdot) induces an isomorphism $\nu : \mathfrak{h} \rightarrow \mathfrak{h}^*$

such that $\alpha_i = \frac{2\nu(\alpha_{\check{i}})}{(\alpha_{\check{i}}, \alpha_{\check{i}})}$ and $\alpha_{\check{i}} = \frac{2\nu^{-1}(\alpha_i)}{(\alpha_i, \alpha_i)}$ for all i .

There exists a totally isotropic subspace \mathfrak{h}'' of \mathfrak{h} (relatively to (\cdot, \cdot)) which is in duality with the center \mathfrak{c} of \mathfrak{g} . In particular, \mathfrak{h}'' defines a supplementary subspace of \mathfrak{h}' in \mathfrak{h} .

Note that any invariant symmetric bilinear form b on \mathfrak{g} satisfying $b(\alpha_{\check{i}}, \alpha_{\check{i}}) > 0$, $\forall i \in I$, is nondegenerate and $b(\alpha_{\check{i}}, h) = \frac{b(\alpha_{\check{i}}, \alpha_{\check{i}})}{2} \langle \alpha_i, h \rangle$, $\forall i \in I, \forall h \in \mathfrak{h}$. It follows that the restriction of b to \mathfrak{g}' is proportional to that of (\cdot, \cdot) . In particular, if A is non-singular, then the invariant bilinear form (\cdot, \cdot) satisfying the condition 1.2 is unique up to a positive rational factor.

1.5. The Tits cone. (See [11], chap. 3 and 5).

Let $C := \{h \in \mathfrak{h}_{\mathbb{R}}; \langle \alpha_i, h \rangle \geq 0, \forall i \in I\}$ be the fundamental chamber (relative to the root basis Π) and let $X := \bigcup_{w \in W} w(C)$ be the Tits cone. We have the following

description of the Tits cone:

- (1) $X = \{h \in \mathfrak{h}_{\mathbb{R}}; \langle \alpha, h \rangle < 0 \text{ only for a finite number of } \alpha \in \Delta^+\}$.
- (2) $X = \mathfrak{h}_{\mathbb{R}}$ if and only if the generalized Cartan matrix A is of finite type.
- (3) If A is of affine type, then $X = \{h \in \mathfrak{h}_{\mathbb{R}}; \langle \delta, h \rangle > 0\} \cup \mathbb{R}\nu^{-1}(\delta)$, where δ is the lowest imaginary positive root of Δ^+ .
- (4) If A is of indefinite type, then the closure of the Tits cone, for the metric topology on $\mathfrak{h}_{\mathbb{R}}$, is $\bar{X} = \{h \in \mathfrak{h}_{\mathbb{R}}; \langle \alpha, h \rangle \geq 0, \forall \alpha \in \Delta_{im}^+\}$.
- (5) If $h \in X$, then h lies in the interior $\overset{\circ}{X}$ of X if and only if the fixator W_h of h , in the Weyl group W , is finite. Thus $\overset{\circ}{X}$ is the union of finite type facets of X .
- (6) If A is hyperbolic, then $\bar{X} \cup (-\bar{X}) = \{h \in \mathfrak{h}_{\mathbb{R}}; (h, h) \leq 0\}$ and the set of imaginary roots is $\Delta^{im} = \{\alpha \in Q \setminus \{0\}; (\alpha, \alpha) \leq 0\}$, where $Q = \mathbb{Z}\Pi$ is the root lattice.

Remark 1.3. Combining (3) and (4) one obtains that if A is not of finite type then $\bar{X} = \{h \in \mathfrak{h}_{\mathbb{R}}; \langle \alpha, h \rangle \geq 0, \forall \alpha \in \Delta_{im}^+\}$.

1.6. Graded Kac-Moody Lie algebras. From now on we suppose that the Kac-Moody Lie algebra \mathfrak{g} is indecomposable and symmetrizable.

Definition 1.4. Let Σ be a root system of Kac-Moody type. The Kac-Moody Lie algebra \mathfrak{g} is said to be finitely Σ -graded if :

- (i) \mathfrak{g} contains, as a subalgebra, a Kac-Moody algebra \mathfrak{m} whose root system relative to Cartan subalgebra \mathfrak{a} is equal to Σ .
- (ii) $\mathfrak{g} = \sum_{\alpha \in \Sigma \cup \{0\}} V_{\alpha}$, with $V_{\alpha} = \{x \in \mathfrak{g}; [a, x] = \langle \alpha, a \rangle x, \forall a \in \mathfrak{a}\}$.
- (iii) V_{α} is finite dimensional for all $\alpha \in \Sigma \cup \{0\}$.

We say that \mathfrak{m} (as in (i) above) is a grading subalgebra, and $(\mathfrak{g}, \Sigma, \mathfrak{m})$ a gradation with finite multiplicities (or, to be short, a finite gradation).

Note that from (ii) the Cartan subalgebra \mathfrak{a} of \mathfrak{m} is $\text{ad}(\mathfrak{g})$ -diagonalizable, and we may assume that \mathfrak{a} is contained in the standard Cartan subalgebra \mathfrak{h} of \mathfrak{g} .

Lemma 1.5. *Let \mathfrak{g} be a Kac-Moody algebra finitely Σ -graded, with grading subalgebra \mathfrak{m} . If \mathfrak{m} itself is finitely Σ' -graded (for some root system Σ' of Kac-Moody type), then \mathfrak{g} is finitely Σ' -graded.*

Proof. If \mathfrak{m}' is the grading subalgebra of \mathfrak{m} , we may suppose the Cartan subalgebras such that $\mathfrak{a}' \subset \mathfrak{a} \subset \mathfrak{h}$, with obvious notations. Conditions (i) and (ii) are clearly satisfied for \mathfrak{g} , \mathfrak{m}' and \mathfrak{a}' . Condition (iii) for \mathfrak{m} and Σ' tells that, for all $\alpha' \in \Sigma'$, the set $\{\alpha \in \Sigma \mid \alpha|_{\mathfrak{a}'} = \alpha'\}$ is finite. But $V_{\alpha'} = \oplus_{\alpha|_{\mathfrak{a}'} = \alpha'} V_{\alpha}$, so each $V_{\alpha'}$ is finite dimensional if this is true for each V_{α} . \square

1.7. Examples of gradations.

- 1) Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ the root system of \mathfrak{g} relative to \mathfrak{h} , then \mathfrak{g} is finitely Δ -graded : this is the trivial gradation of \mathfrak{g} by its own root system.
- 2) Let $\mathfrak{g}_{\mathbb{R}}$ be an almost split real form of \mathfrak{g} (see [2]) and let $\mathfrak{t}_{\mathbb{R}}$ be a maximal split toral subalgebra of $\mathfrak{g}_{\mathbb{R}}$. Suppose that the restricted root system $\Delta' = \Delta(\mathfrak{g}_{\mathbb{R}}, \mathfrak{t}_{\mathbb{R}})$ is

reduced of Kac-Moody type. In [[4], §9], N. Bardy constructed a split real Kac-Moody subalgebra $\mathfrak{l}_{\mathbb{R}}$ of $\mathfrak{g}_{\mathbb{R}}$ such that $\Delta' = \Delta(\mathfrak{l}_{\mathbb{R}}, \mathfrak{t}_{\mathbb{R}})$, then \mathfrak{g} is obviously finitely Δ' -graded.

We get thus many examples coming from known tables for almost split real forms: see [2] in the affine case and [6] in the hyperbolic case.

3) When $\mathfrak{g}_{\mathbb{R}}$ is an almost compact real form of \mathfrak{g} , the same constructions should lead to gradations by finite root systems, as in [5] e.g..

2. GRADATIONS ASSOCIATED TO C -ADMISSIBLE PAIRS.

We recall some definitions introduced by H. Rubenthaler ([19]) and J. Nervi ([15], [16]). Let J be a subset of I of finite type. For $k \in I \setminus J$, we denote by I_k the connected component, containing k , of the Dynkin subdiagram corresponding to $J \cup \{k\}$, and let $J_k := I_k \setminus \{k\}$.

Suppose now that I_k is of finite type for all $k \in I \setminus J$: that is always the case if \mathfrak{g} is of affine type and $|I \setminus J| \geq 2$ or \mathfrak{g} is of hyperbolic type and $|I \setminus J| \geq 3$. For $k \in I \setminus J$, let $\mathfrak{g}(I_k)$ be the simple subalgebra generated by $\mathfrak{g}_{\pm\alpha_i}$, $i \in I_k$, then $\mathfrak{h}_{I_k} = \mathfrak{h} \cap \mathfrak{g}(I_k) = \sum_{i \in I_k} \mathbb{C}\alpha_i$ is a Cartan subalgebra of $\mathfrak{g}(I_k)$. Let H_k be the unique element of \mathfrak{h}_{I_k} such that $\langle \alpha_i, H_k \rangle = 2\delta_{i,k}$, $\forall i \in I_k$.

Definition 2.1. We preserve the notations and the assumptions introduced above.

1) Let $k \in I \setminus J$.

(i) The pair (I_k, J_k) is called admissible if there exist $E_k, F_k \in \mathfrak{g}(I_k)$ such that (E_k, H_k, F_k) is an \mathfrak{sl}_2 -triple.

(ii) The pair (I_k, J_k) is called C -admissible if it is admissible and the simple Lie algebra $\mathfrak{g}(I_k)$ is A_1 -graded by the root system of type A_1 associated to the \mathfrak{sl}_2 -triple (E_k, H_k, F_k) .

2) the pair (I, J) is called C -admissible if the pairs (I_k, J_k) are C -admissible for all $k \in I \setminus J$.

Schematically, any C -admissible pair (I, J) is represented by the Dynkin diagram, corresponding to A , on which the vertices indexed by J are denoted by white circles \circ and those of $I \setminus J$ are denoted by black circles \bullet .

It is known that, when (I_k, J_k) is admissible, $H_k = \sum_{i \in I_k} n_{i,k} \alpha_i$, where $n_{i,k}$ are positive integers (see [19] or [[16]; Prop. 1.4.1.2]).

Remark 2.2. Note that this definition, for C -admissible pairs, is equivalent to that introduced by Rubenthaler and Nervi (see [19], [15]) in terms of prehomogeneous spaces of parabolic type: if (I_k, J_k) is C -admissible, define for $p \in \mathbb{Z}$, the subspace $d_{k,p} := \{X \in \mathfrak{g}(I_k); [H_k, X] = 2pX\}$; then $(d_{k,0}, d_{k,1})$ is an irreducible regular and commutative prehomogeneous space of parabolic type, and $d_{k,p} = \{0\}$ for $|p| \geq 2$. Then we say that (I_k, J_k) is an irreducible C -admissible pair. According to Rubenthaler and Nervi ([19]; table1) or [[15]; table 2]) the irreducible C -admissible pair (I_k, J_k) should be among the list in Table 1 below.

Definition 2.3. Let J be a subset of I and let $i, k \in I \setminus J$. We say that i and k are J -connected relative to A if there exist $j_0, j_1, \dots, j_{p+1} \in I$ such that $j_0 = i$, $j_{p+1} = k$, $j_s \in J$, $\forall s = 1, 2, \dots, p$, and $a_{j_s, j_{s+1}} \neq 0$, $\forall s = 0, 1, \dots, p$.

Remark 2.4. Note that the relation “to be J -connected” is symmetric on i and k . As the generalized Cartan matrix is assumed to be indecomposable, for any

vertices $i, k \in I \setminus J$ there exist $i_0, i_1, \dots, i_{p+1} \in I \setminus J$ such that $i_0 = i$, $i_{p+1} = k$ and i_s and i_{s+1} are J -connected for all $s = 0, 1, \dots, p$.

Table 1

List of irreducible C -admissible pairs

$A_{2n-1}, n \geq 1$	
$B_n, n \geq 3$	
$C_n, n \geq 2$	
$D_{n,1}, n \geq 4$	
$D_{2n,2}, n \geq 2$	
E_7	

Let us assume from now on that (I, J) is a C -admissible pair and let $I' := I \setminus J$. For $k \in I'$, let (E_k, H_k, F_k) be an \mathfrak{sl}_2 -triple associated to the irreducible C -admissible pair (I_k, J_k) .

Lemma 2.5. *Let $k \neq l \in I'$, then :*

- 1) $\langle \alpha_l, H_k \rangle \in \mathbb{Z}^-$.
- 2) *the following assertions are equivalent :*
 - i) k, l are J -connected
 - ii) $\langle \alpha_l, H_k \rangle$ is a negative integer
 - iii) $\langle \alpha_k, H_l \rangle$ is a negative integer

Proof.

1) Recall that $H_k = \sum_{i \in I_k} n_{i,k} \alpha_i^\vee$, where $n_{i,k}$ are positive integers. As $l \notin I_k$, we have that $\langle \alpha_l, H_k \rangle = \sum_{i \in I_k} n_{i,k} \langle \alpha_l, \alpha_i^\vee \rangle \in \mathbb{Z}^-$.

2) In view of Remark 2.4, it suffices to prove the equivalence between i) and ii). Since I_k is the connected component of $J \cup \{k\}$ containing k , the assertion i) is equivalent to say that the vertex l is connected to I_k , so there exists $i_k \in I_k$ such that $\langle \alpha_l, \alpha_{i_k}^\vee \rangle < 0$ and hence $\langle \alpha_l, H_k \rangle < 0$. \square

Proposition 2.6. *Let $\mathfrak{h}^J = \Pi_J^\perp = \{h \in \mathfrak{h}, \langle \alpha_j, h \rangle = 0, \forall j \in J\}$. For $k \in I'$, denote by $\alpha'_k = \alpha_k|_{\mathfrak{h}^J}$ the restriction of α_k to the subspace \mathfrak{h}^J of \mathfrak{h} , and $\Pi^J = \{\alpha'_k; k \in I'\}$, $\Pi^{J^\vee} = \{H_k; k \in I'\}$. For $k, l \in I'$, put $a'_{k,l} = \langle \alpha_l, H_k \rangle$ and*

$A^J = (a'_{k,l})_{k,l \in I'}$. Then A^J is an indecomposable and symmetrizable generalized Cartan matrix, $(\mathfrak{h}^J, \Pi^J, \Pi^{J\vee})$ is a realization of A^J and $\text{corank}(A^J) = \text{corank}(A)$.

Proof. The fact that $a'_{k,k} = 2$ follows from the definition of H_k for $k \in I'$. If $k \neq l \in I'$, then by lemma 2.5, $a'_{k,l} \in \mathbb{Z}^-$ and $a'_{k,l} \neq 0$ if and only if $a'_{l,k} \neq 0$. Hence A^J is a generalized Cartan matrix. As the matrix A is indecomposable, A_J is also indecomposable (see Remark 2.4). Clearly $\Pi^J = \{\alpha'_k; k \in I'\}$ is free in \mathfrak{h}^{J*} the dual space of \mathfrak{h}^J , $\Pi^{J\vee} = \{H_k; k \in I'\}$ is free in \mathfrak{h}^J and by construction $\langle \alpha_l, H_k \rangle = a'_{k,l}$, $\forall k, l \in I'$.

We have to prove that $\dim(\mathfrak{h}^J) - |I'| = \text{corank}(A^J)$. As J is of finite type, the restriction of the invariant bilinear form (\cdot, \cdot) to \mathfrak{h}_J is nondegenerate and \mathfrak{h}_J is contained in $\mathfrak{h}' = \bigoplus_{i \in I} \mathbb{C}\alpha_i$. Therefore

$$\mathfrak{h} = \mathfrak{h}^J \oplus^\perp \mathfrak{h}_J$$

and

$$\mathfrak{h}' = (\mathfrak{h}' \cap \mathfrak{h}^J) \oplus \mathfrak{h}_J.$$

It follows that $\dim(\mathfrak{h}' \cap \mathfrak{h}^J) = |I'| = \dim(\bigoplus_{k \in I'} \mathbb{C}H_k)$. As the subspace $\bigoplus_{k \in I'} \mathbb{C}H_k$ is contained in $\mathfrak{h}' \cap \mathfrak{h}^J$, we deduce that $\mathfrak{h}' \cap \mathfrak{h}^J = \bigoplus_{k \in I'} \mathbb{C}H_k$. Note that any supplementary subspace $\mathfrak{h}^{J''}$ of $\mathfrak{h}' \cap \mathfrak{h}^J$ in \mathfrak{h}^J is also a supplementary of \mathfrak{h}' in \mathfrak{h} ; hence, we have that $\text{corank}(A) = \dim(\mathfrak{h}^{J''}) = \dim(\mathfrak{h}^J) - |I'|$. In addition, it is known that $\text{corank}(A) = \dim(\mathfrak{c})$, where $\mathfrak{c} = \bigcap_{i \in I} \ker(\alpha_i)$ is the center of \mathfrak{g} . Let $\mathfrak{c}^J = \bigcap_{k \in I'} \ker(\alpha'_k)$, then $\mathfrak{c}^J = \mathfrak{c}$ and $\text{corank}(A^J) = \dim(\mathfrak{c}^J) = \text{corank}(A) = \dim(\mathfrak{h}^J) - |I'|$.

It remains to prove that A^J is symmetrizable. For $k \in I'$, let R_k^J be the fundamental reflection of \mathfrak{h}^J such that $R_k^J(h) = h - \langle \alpha'_k, h \rangle H_k$, $\forall h \in \mathfrak{h}^J$. Let W^J be the Weyl group of A^J generated by R_k^J , $k \in I'$. Let $(\cdot, \cdot)^J$ be the restriction to \mathfrak{h}^J of the invariant bilinear form (\cdot, \cdot) on \mathfrak{h} . Then $(\cdot, \cdot)^J$ is a nondegenerate symmetric bilinear form on \mathfrak{h}^J which is W^J -invariant (see the lemma hereafter). From the relation $(R_k^J(H_k), R_k^J(H_l))^J = (H_k, H_l)^J$ one can deduce that :

$$(H_k, H_l)^J = \frac{(H_k, H_k)^J}{2} a'_{l,k}, \quad \forall k, l \in I',$$

but $(H_k, H_k)^J > 0$, $\forall k \in I'$; hence ${}^t A^J$ (and so A^J) is symmetrizable. \square

Lemma 2.7. For $k \in I' := I \setminus J$, let w_k^J be the longest element of the Weyl group $W(I_k)$ generated by the fundamental reflections r_i , $i \in I_k$. Then w_k^J stabilizes \mathfrak{h}^J and induces the fundamental reflection R_k^J of \mathfrak{h}^J associated to H_k .

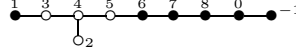
Proof. If one looks at the list above of the irreducible C -admissible pairs, one can see that $w_k^J(\alpha_k) = -\alpha_k$ and that $-w_k^J$ permutes the α_j , $j \in J_k$. In addition, $w_k^J(\alpha_j) = \alpha_j$, $\forall j \in J \setminus J_k$. Now it's clear that w_k^J stabilizes \mathfrak{h}_J and hence it stabilizes $\mathfrak{h}^J = \mathfrak{h}_J^\perp$. Note that $-w_k^J(H_k) \in \mathfrak{h}_{I_k}$ and satisfies the same equations defining H_k . Hence $-w_k^J(H_k) = H_k = -R_k^J(H_k)$. Clearly w_k^J and R_k^J fix both $\ker(\alpha'_k) = \ker(\alpha_k) \cap (\bigcap_{j \in J} \ker(\alpha_j))$. As $\mathfrak{h}^J = \ker(\alpha'_k) \oplus \mathbb{C}H_k$, the reflection R_k^J and W_k^J coincide on \mathfrak{h}^J . \square

Remark 2.8. Actually we can now rediscover the list of irreducible C -admissible pairs given in Remark 2.2. The black vertex k should be invariant under $-w_k^J$ and the corresponding coefficient of the highest root of I_k should be 1 (an easy consequence of the definition 2.1 1) (ii)).

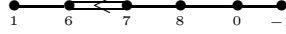
Example 2.9. Consider the hyperbolic generalized Cartan matrix A of type $HE_8^{(1)} = E_{10}$ indexed by $I = \{-1, 0, 1, \dots, 8\}$.

The following two choices for J define C -admissible pairs :

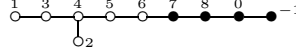
1) $J = \{2, 3, 4, 5\}$.



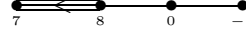
The corresponding generalized Cartan matrix A^J is hyperbolic of type $HF_4^{(1)}$:



2) $J = \{1, 2, 3, 4, 5, 6\}$.



The corresponding generalized Cartan matrix A^J is hyperbolic of type $HG_2^{(1)}$:



Note that the first example corresponds to an almost split real form of the Kac-Moody Lie algebra $\mathfrak{g}(A)$ and A^J is the generalized Cartan matrix associated to the corresponding (reduced) restricted root system (see [6]) whereas the second example does not correspond to an almost split real form of $\mathfrak{g}(A)$.

Lemma 2.10. For $k \in I'$, set $\mathfrak{s}(k) = \mathbb{C}E_k \oplus \mathbb{C}H_k \oplus \mathbb{C}F_k$. Then, the Kac-Moody algebra \mathfrak{g} is an integrable $\mathfrak{s}(k)$ -module via the adjoint representation of $\mathfrak{s}(k)$ on \mathfrak{g} .

Proof. Note that $\mathfrak{s}(k)$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ with standard basis (E_k, H_k, F_k) . It is clear that $\text{ad}(H_k)$ is diagonalizable on \mathfrak{g} and $E_k = \sum_{\alpha} e_{\alpha} \in d_{k,1}$, where α runs over the set $\Delta_{k,1} = \{\alpha \in \Delta(I_k); \langle \alpha, H_k \rangle = 2\}$, $e_{\alpha} \in \mathfrak{g}_{\alpha}$ for $\alpha \in \Delta(I_k)$, and $d_{k,1} := \{X \in \mathfrak{g}(I_k); [H_k, X] = 2X\}$. Since $\Delta_{k,1} \subset \Delta^{re}$, $\text{ad}(e_{\alpha})$ is locally nilpotent for $\alpha \in \Delta_{k,1}$. As $d_{k,1}$ is commutative (see Remark 2.2) we deduce that $\text{ad}(E_k)$ is locally nilpotent on \mathfrak{g} . The same argument shows that $\text{ad}(F_k)$ is also locally nilpotent. Hence, the Kac-Moody algebra \mathfrak{g} is an integrable $\mathfrak{s}(k)$ -module. \square

Proposition 2.11. Let \mathfrak{g}^J be the subalgebra of \mathfrak{g} generated by \mathfrak{h}^J and $E_k, F_k, k \in I'$. Then \mathfrak{g}^J is the Kac-Moody Lie algebra associated to the realization $(\mathfrak{h}^J, \Pi^J, \Pi^{J^\vee})$ of the generalized Cartan matrix A^J .

Proof. It is not difficult to check that the following relations hold in the Lie subalgebra \mathfrak{g}^J :

$$\begin{aligned} [\mathfrak{h}^J, \mathfrak{h}^J] &= 0, & [E_k, F_l] &= \delta_{k,l} H_k & (k, l \in I'); \\ [h, E_k] &= \langle \alpha'_k, h \rangle E_k, & [h, F_k] &= -\langle \alpha'_k, h \rangle F_k & (h \in \mathfrak{h}^J, k \in I'). \end{aligned}$$

We have to prove the Serre's relations :

$$(\text{ad} E_k)^{1-a'_{k,l}}(E_l) = 0, \quad (\text{ad} F_k)^{1-a'_{k,l}}(F_l) = 0 \quad (k \neq l \in I').$$

For $k \in I'$, let $\mathfrak{s}(k) = \mathbb{C}F_k \oplus \mathbb{C}H_k \oplus \mathbb{C}E_k$ be the Lie subalgebra of \mathfrak{g} isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. Let $l \neq k \in I'$; note that $[H_k, F_l] = -a'_{k,l} F_l$ and $[E_k, F_l] = 0$, which means

that F_l is a primitive weight vector for $\mathfrak{s}(k)$. As \mathfrak{g} is an integrable $\mathfrak{s}(k)$ -module (see Lemma 2.10) the primitive weight vector F_l is contained in a finite dimensional $\mathfrak{s}(k)$ -submodule (see [11]; 3.6). The relation $(\text{ad} F_k)^{1-a'_{k,l}}(F_l) = 0$ follows from the representation theory of $\mathfrak{sl}_2(\mathbb{C})$ (see [11]; 3.2). By similar arguments we prove that $(\text{ad} E_k)^{1-a'_{k,l}}(E_l) = 0$.

Now \mathfrak{g}^J is a quotient of the Kac-Moody algebra associated to A^J and $(\mathfrak{h}^J, \Pi^J, \Pi^{J\vee})$. By [11, 1.7] it is equal to it. \square

Definition 2.12. The Kac-Moody Lie algebra \mathfrak{g}^J is called the C -admissible algebra associated to the C -admissible pair (I, J) .

Proposition 2.13. The Kac-Moody algebra \mathfrak{g} is an integrable \mathfrak{g}^J -module with finite multiplicities.

Proof. The \mathfrak{g}^J -module \mathfrak{g} is clearly $\text{ad}(\mathfrak{h}^J)$ -diagonalizable and $\text{ad}(E_k)$, $\text{ad}(F_k)$ are locally nilpotent on \mathfrak{g} for $k \in I'$ (see Lemma 2.10). Hence, \mathfrak{g} is an integrable \mathfrak{g}^J -module. For $\alpha \in \Delta$, let $\alpha' = \alpha|_{\mathfrak{h}^J}$ be the restriction of α to \mathfrak{h}^J . Set $\Delta' = \{\alpha'; \alpha \in \Delta\} \setminus \{0\}$. Then the set of weights, for the \mathfrak{g}^J -module \mathfrak{g} , is exactly $\Delta' \cup \{0\}$. Note that for $\alpha \in \Delta$, $\alpha' = 0$ if and only if $\alpha \in \Delta(J)$. In particular, the weight space $V_0 = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta(J)} \mathfrak{g}_\alpha)$ corresponding to the null weight is finite dimensional.

Let $\alpha = \sum_{i \in I} n_i \alpha_i \in \Delta$ such that $\alpha' \neq 0$. We will see that the corresponding weight

space $V_{\alpha'}$ is finite dimensional. Note that $V_{\alpha'} = \bigoplus_{\beta' = \alpha'} \mathfrak{g}_{\beta'}$. Let $\beta = \sum_{i \in I} m_i \alpha_i \in \Delta$

such that $\beta' = \alpha' = \sum_{k \in I'} n_k \alpha'_k$, then $m_k = n_k$, $\forall k \in I'$, since $\Pi^J = \{\alpha'_k, k \in I'\}$

is free in $(\mathfrak{h}^J)^*$. In particular, β and α are of the same sign, and we may assume $\alpha \in \Delta^+$. Let $ht_J(\beta) = \sum_{j \in J} m_j$ be the height of β relative to J , and let W_J be

the finite subgroup of W generated by r_j , $j \in J$. Since W_J fixes pointwise \mathfrak{h}^J , we deduce that $\gamma' = \beta'$, $\forall \gamma \in W_J \beta$, and so we may assume that $ht_J(\beta)$ is minimal among the roots in $W_J \beta$. From the inequality $ht_J(\beta) \leq ht_J(r_j(\beta))$, $\forall j \in J$, we get $\langle \beta, \alpha_j \rangle \leq 0$, $\forall j \in J$. Let $\rho_{\check{J}}$ be the half sum of positive coroots of $\Delta(J)$. It is known that $\langle \alpha_j, \rho_{\check{J}} \rangle = 1$, $\forall j \in J$. It follows that $0 \geq \langle \beta, \rho_{\check{J}} \rangle = \sum_{j \in J} m_j + \sum_{k \in I'} n_k \langle \alpha_k, \rho_{\check{J}} \rangle$, so

finally, we obtain : $ht_J(\beta) \leq \sum_{k \in I'} -n_k \langle \alpha_k, \rho_{\check{J}} \rangle$. Hence, there is just a finite number

of possibilities for β , and then α' is of finite multiplicity. \square

Theorem 2.14. Let Δ^J be the root system of the pair $(\mathfrak{g}^J, \mathfrak{h}^J)$, then the Kac-Moody Lie algebra \mathfrak{g} is finitely Δ^J -graded, with grading subalgebra \mathfrak{g}^J .

Proof. Let $\Delta' = \{\alpha', \alpha \in \Delta\} \setminus \{0\}$ be the set of nonnull weights of the \mathfrak{g}^J -module \mathfrak{g} relative to \mathfrak{h}^J . Let $\Delta'_+ = \{\alpha' \in \Delta', \alpha \in \Delta^+\}$ and Δ_+^J the set of positive roots of Δ^J relative to the root basis Π^J . We have to prove that $\Delta' = \Delta^J$ or equivalently $\Delta'_+ = \Delta_+^J$. Let $Q^J = \mathbb{Z}\Pi^J$ be the root lattice of Δ^J and $Q_+^J = \mathbb{Z}^+\Pi^J$. It is known that the positive root system Δ_+^J is uniquely defined by the following properties (see [11], Ex. 5.4) :

- (i) $\Pi^J \subset \Delta_+^J \subset Q_+^J$, $2\alpha'_i \notin \Delta_+^J$, $\forall i \in I'$;
- (ii) if $\alpha' \in \Delta_+^J$, $\alpha' \neq \alpha'_i$, then the set $\{\alpha' + k\alpha'_i; k \in \mathbb{Z}\} \cap \Delta_+^J$ is a string

$\{\alpha' - p\alpha'_i, \dots, \alpha' + q\alpha'_i\}$, where $p, q \in \mathbb{Z}^+$ and $p - q = \langle \alpha', H_i \rangle$;
 (iii) if $\alpha' \in \Delta_+^J$, then $\text{supp}(\alpha')$ is connected.

We will see that Δ'_+ satisfies these three properties and hence $\Delta'_+ = \Delta_+^J$. Clearly $\Pi^J \subset \Delta'_+ \subset Q_+^J$. For $\alpha \in \Delta$ and $k \in I'$, the condition $\alpha' \in \mathbb{N}\alpha_k$ implies $\alpha \in \Delta(I_k)^+$. As (I_k, J_k) is C -admissible for $k \in I'$, the highest root of $\Delta(I_k)^+$ has coefficient 1 on the root α_k (cf. Remark 2.8). It follows that $2\alpha'_k \notin \Delta'_+$ and (i) is satisfied. By Proposition 2.13, \mathfrak{g} is an integrable \mathfrak{g}^J -module with finite multiplicities. Hence, the propriety (ii) follows from [[11]; prop.3.6]. Let $\alpha \in \Delta_+$, then $\text{supp}(\alpha)$ is connected and $\text{supp}(\alpha') \subset \text{supp}(\alpha)$. Let $k, l \in \text{supp}(\alpha')$; if k, l are J -connected in $\text{supp}(\alpha)$ relative to the generalized Cartan matrix A (cf. 2.3), then by lemma 2.5, k, l are linked in I' relative to the generalized Cartan matrix A^J . Hence, the connectedness of $\text{supp}(\alpha')$, relative to A^J , follows from that of $\text{supp}(\alpha)$ relative to A (see Remark 2.4) and (iii) is satisfied. \square

Remark 2.15. Note that the definition of C -admissible pair can be extended to decomposable Kac-Moody Lie algebras : thus if I^1, I^2, \dots, I^m are the connected components of I and $J^k = J \cap I^k$, $k = 1, 2, \dots, m$, then (I, J) is C -admissible if and only if (I^k, J^k) is for all $k = 1, 2, \dots, m$. In particular, the corresponding C -admissible algebra is $\mathfrak{g}^J = \bigoplus_{k=1}^m \mathfrak{g}(I^k)^{J^k}$, where $\mathfrak{g}(I^k)^{J^k}$ is the C -admissible subalgebra of $\mathfrak{g}(I^k)$ corresponding to (I^k, J^k) , $k = 1, 2, \dots, m$.

3. GENERAL GRADATIONS.

Let \mathfrak{m} be an indecomposable Kac-Moody subalgebra of \mathfrak{g} and let \mathfrak{a} be a Cartan subalgebra of \mathfrak{m} . Put $\Sigma = \Delta(\mathfrak{m}, \mathfrak{a})$ the corresponding root system. We assume that $\mathfrak{a} \subset \mathfrak{h}$ and that \mathfrak{g} is finitely Σ -graded with \mathfrak{m} as grading subalgebra. Thus $\mathfrak{g} = \sum_{\gamma \in \Sigma \cup \{0\}} V_\gamma$, with $V_\gamma = \{x \in \mathfrak{g}; [a, x] = \langle \gamma, a \rangle x, \forall a \in \mathfrak{a}\}$ is finite dimensional for all $\gamma \in \Sigma \cup \{0\}$. For $\alpha \in \Delta$, denote by $\rho_a(\alpha)$ the restriction of α to \mathfrak{a} . As \mathfrak{g} is Σ -graded, one has $\rho_a(\Delta \cup \{0\}) = \Sigma \cup \{0\}$.

Lemma 3.1.

- 1) Let \mathfrak{c} be the center of \mathfrak{g} and denote by \mathfrak{c}_a the center of \mathfrak{m} . Then $\mathfrak{c}_a = \mathfrak{c} \cap \mathfrak{a}$. In particular, if \mathfrak{g} is perfect, then the grading subalgebra \mathfrak{m} is also perfect.
- 2) Suppose that $\Delta^{im} \neq \emptyset$, then $\rho_a(\Delta^{im}) \subset \Sigma^{im}$.

Proof.

- 1) It is clear that $\mathfrak{c} \cap \mathfrak{a} \subset \mathfrak{c}_a$. Since \mathfrak{g} is Σ -graded, we deduce that \mathfrak{c}_a is contained in the center \mathfrak{c} of \mathfrak{g} , hence $\mathfrak{c}_a \subset \mathfrak{c} \cap \mathfrak{a}$. If \mathfrak{g} is perfect, then $\mathfrak{g} = \mathfrak{g}'$, $\mathfrak{h} = \mathfrak{h}'$, $\mathfrak{c} = \{0\}$; so $\mathfrak{c}_a = \{0\}$, $\mathfrak{a} = \mathfrak{a}'$ and $\mathfrak{m} = \mathfrak{m}'$.
- 2) If $\alpha \in \Delta^{im}$, then $\mathbb{N}\alpha \subset \Delta$. Since V_0 is finite dimensional, $\rho_a(\alpha) \neq 0$ and $\mathbb{N}\rho_a(\alpha) \subset \Sigma$, hence $\rho_a(\alpha) \in \Sigma^{im}$. \square

In the following, we will show that the Kac-Moody Lie algebra \mathfrak{g} and the grading subalgebra \mathfrak{m} are of the same type.

Lemma 3.2. *The Kac-Moody Lie algebra \mathfrak{g} is of indefinite type if and only if Δ^{im} generates the dual space $(\mathfrak{h}/\mathfrak{c})^*$ of $\mathfrak{h}/\mathfrak{c}$.*

Proof. Note that the root basis $\Pi = \{\alpha_i, i \in I\}$ induces a basis of the vector space $(\mathfrak{h}/\mathfrak{c})^*$. In particular, $\dim(\mathfrak{h}/\mathfrak{c})^* \geq 2$ when Δ^{im} is nonempty. Suppose now that \mathfrak{g} is of indefinite type. Let $\alpha \in \Delta_+^{sim}$ be a positive strictly imaginary root satisfying $\langle \alpha, \alpha_i \rangle < 0, \forall i \in I$; then, $r_i(\alpha) = \alpha - \langle \alpha, \alpha_i \rangle \alpha_i \in \Delta_+^{im}$ for all $i \in I$. In particular, the vector subspace $\langle \Delta^{im} \rangle$ spanned by Δ^{im} contains Π and hence is equal to $(\mathfrak{h}/\mathfrak{c})^*$. Conversely, if Δ^{im} generates $(\mathfrak{h}/\mathfrak{c})^*$, then Δ^{im} is nonempty and contains at least two linearly independent imaginary roots; hence Δ can not be of finite or affine type. \square

Proposition 3.3.

- 1) The Kac-Moody Lie Algebra \mathfrak{g} and the grading subalgebra \mathfrak{m} are of the same type.
- 2) Suppose \mathfrak{g} of indefinite type and Lorentzian, then \mathfrak{m} is also Lorentzian.

Proof.

- 1) If \mathfrak{g} is of finite type, then Δ is finite and hence $\Sigma = \rho_a(\Delta) \setminus \{0\}$ is finite.

If \mathfrak{g} is affine, let δ be the lowest positive imaginary root. One can choose a root basis $\Pi_a = \{\gamma_i, i \in \bar{I}\}$ of Σ so that $\bar{\delta} := \rho_a(\delta)$ is a positive imaginary root. Note that $\mathfrak{a}' := \mathfrak{a} \cap \mathfrak{m}' \subset \mathfrak{h}'$; in particular $\bar{\delta}(\mathfrak{a}') = \{0\}$ and $\langle \bar{\delta}, \gamma_i \rangle = 0, \forall i \in \bar{I}$. It follows that \mathfrak{m} is affine (see [[11]; Prop. 4.3]).

Suppose now that \mathfrak{g} is of indefinite type. Thanks to Lemma 3.2, it suffices to prove that Σ^{im} generates $(\mathfrak{a}/\mathfrak{c}_a)^*$, where $\mathfrak{c}_a = \mathfrak{c} \cap \mathfrak{a}$ is the center of \mathfrak{m} . The natural homomorphism of vector spaces $\pi : \mathfrak{a} \rightarrow \mathfrak{h}/\mathfrak{c}$ induces a monomorphism $\bar{\pi} : \mathfrak{a}/\mathfrak{c}_a \rightarrow \mathfrak{h}/\mathfrak{c}$. By duality, the homomorphism $\bar{\pi}^* : (\mathfrak{h}/\mathfrak{c})^* \rightarrow (\mathfrak{a}/\mathfrak{c}_a)^*$ is surjective and $\bar{\pi}^*(\Delta^{im}) \subset \Sigma^{im}$ generates $(\mathfrak{a}/\mathfrak{c}_a)^*$.

- 2) Suppose that \mathfrak{g} is Lorentzian and let (\cdot, \cdot) be an invariant nondegenerate bilinear form on \mathfrak{g} . Then, the restriction of (\cdot, \cdot) to $\mathfrak{h}_{\mathbb{R}}$ has signature $(++ \dots +, -)$ and any maximal totally isotropic subspace of $\mathfrak{h}_{\mathbb{R}}$ relatively to (\cdot, \cdot) is one dimensional. Let $\mathfrak{a}_{\mathbb{R}} := \mathfrak{a} \cap \mathfrak{h}_{\mathbb{R}}$ and let $(\cdot, \cdot)_a$ be the restriction of (\cdot, \cdot) to \mathfrak{m} . As \mathfrak{m} is of indefinite type, $\dim(\mathfrak{a}) \geq 2$ and the restriction of $(\cdot, \cdot)_a$ to $\mathfrak{a}_{\mathbb{R}}$ is nonnull. It follows that the orthogonal subspace \mathfrak{m}^{\perp} of \mathfrak{m} relatively to $(\cdot, \cdot)_a$ is a proper ideal of \mathfrak{m} . Since \mathfrak{m} is perfect (because \mathfrak{g} is) we deduce that $\mathfrak{m}^{\perp} = \{0\}$ (cf. [11, 1.7]) and the invariant bilinear form $(\cdot, \cdot)_a$ is nondegenerate. It follows that \mathfrak{m} is symmetrizable and the restriction of $(\cdot, \cdot)_a$ to \mathfrak{a} is nondegenerate. As \mathfrak{m} is of indefinite type, the restriction of $(\cdot, \cdot)_a$ to $\mathfrak{a}_{\mathbb{R}}$ can not be positive definite. Hence, the bilinear form $(\cdot, \cdot)_a$ has signature $(++ \dots +, -)$ on $\mathfrak{a}_{\mathbb{R}}$ and then the grading subalgebra \mathfrak{m} is Lorentzian. \square

Definition 3.4. Let Π_a be a root basis of Σ and let Σ^+ be the corresponding set of positive roots. The root basis is said to be adapted to the root basis Π of Δ if $\rho_a(\Delta^+) \subset \Sigma^+ \cup \{0\}$.

We will see that adapted root bases always exist.

Definition 3.5. ([3]; 5.2.6) Suppose that $\Delta^{im} \neq \emptyset$. Let $\alpha, \beta \in \Delta^{im}$.

- (i) The imaginary roots α and β are said to be linked if $\mathbb{N}\alpha + \mathbb{N}\beta \subset \Delta$ or $\beta \in \mathbb{Q}^+\alpha$.
- (ii) The imaginary roots α and β are said to be linkable if there exists a finite family of imaginary roots $(\beta_i)_{0 \leq i \leq n+1}$ such that $\beta_0 = \alpha, \beta_{n+1} = \beta$ and β_i and β_{i+1} are linked for all $i = 0, 1, \dots, n$.

Proposition 3.6. ([3]; Prop. 5.2.7) Suppose that $\Delta^{im} \neq \emptyset$. To be linkable is an equivalence relation on Δ^{im} and, if A is indecomposable, there exist exactly two equivalence classes : Δ_-^{im} and Δ_+^{im} .

Lemma 3.7. *Suppose that $\Delta^{im} \neq \emptyset$, then there exists a root basis of Σ such that $\rho_a(\Delta_+^{im}) \subset \Sigma_+^{im}$.*

Proof. Let $\alpha, \beta \in \Delta_+^{im}$, then, by Proposition 3.6, α and β are linkable and so are $\rho_a(\alpha)$ and $\rho_a(\beta)$. Since Σ is assumed to be indecomposable, $\rho_a(\alpha)$ and $\rho_a(\beta)$ are of the same sign. One can choose a root basis of Σ such that $\rho_a(\alpha)$ and $\rho_a(\beta)$ are positive, and then we have $\rho_a(\Delta_+^{im}) \subset \Sigma_+^{im}$. \square

Corollary 3.8. *Let Π_a be a root basis of Σ such that $\rho_a(\Delta_+^{im}) \subset \Sigma_+^{im}$ and let X_a be the corresponding positive Tits cone. Then we have $\bar{X}_a \subset \bar{X} \cap \mathfrak{a}$.*

Proof. As $\Delta^{im} \neq \emptyset$, one has $\bar{X} = \{h \in \mathfrak{h}_{\mathbb{R}}; \langle \alpha, h \rangle \geq 0, \forall \alpha \in \Delta_+^{im}\}$ (see Remark 1.3). The corollary follows from Lemma 3.7. \square

Lemma 3.9. *Suppose that $\Delta^{im} \neq \emptyset$. Let $p \in \bar{X}$ such that $\langle \alpha, p \rangle \in \mathbb{Z}$, $\forall \alpha \in \Delta$, and $\langle \beta, p \rangle > 0$, $\forall \beta \in \Delta_+^{im}$. Then $p \in \bar{X}$.*

Proof.

The result is clear when Δ is of affine type since $\bar{X} = \bar{X}^\circ = \{h \in \mathfrak{h}_{\mathbb{R}}; \langle \delta, h \rangle > 0\}$. Suppose now that Δ is of indefinite type. If one looks to the proof of Proposition 5.8.c) in [11], one can show that an element $p \in \bar{X}$ satisfying the conditions of the lemma lies in X . As Δ_+^{im} is W -invariant, we may assume that p lies in the fundamental chamber C . Hence there exists a subset J of I such that $\{\alpha \in \Delta; \langle \alpha, p \rangle = 0\} = \Delta_J = \Delta \cap \sum_{j \in J} \mathbb{Z}\alpha_j$. Since $\Delta_J \cap \Delta^{im} = \emptyset$, the root subsystem Δ_J is of finite type and p lies in the finite type facet of type J . \square

Theorem 3.10. *There exists a root basis Π_a of Σ which is adapted to the root basis Π of Δ . Moreover, there exists a finite type subset J of I such that $\Delta_J = \{\alpha \in \Delta; \rho_a(\alpha) = 0\}$.*

N.B. This is part 1) of Theorem 2.

Proof. Let $\Pi_a = \{\gamma_i, i \in \bar{I}\}$ be a root basis of Σ such that $\rho_a(\Delta_+^{im}) \subset \Sigma_+^{im}$, where \bar{I} is just a set indexing the basis elements. Let $p \in \mathfrak{a}$ such that $\langle \gamma_i, p \rangle = 1$, $\forall i \in \bar{I}$ and let $P = \{\alpha \in \Delta; \langle \alpha, p \rangle \geq 0\}$. If Δ is finite, then P is clearly a parabolic subsystem of Δ and the result is trivial. Suppose now that $\Delta^{im} \neq \emptyset$; then p satisfies the conditions of the Lemma 3.9 and we may assume that p lies in the facet of type J for some subset J of finite type in I . In which case $P = \Delta_J \cup \Delta^+$ is the standard parabolic subsystem of finite type J . Note that, for $\gamma \in \Sigma^+$, one has $\langle \gamma, p \rangle = ht_a(\gamma)$ the height of γ with respect to Π_a . It follows that $\{\alpha \in \Delta; \rho_a(\alpha) = 0\} = \Delta_J$, in particular, $\rho_a(\Delta^+) = \rho_a(P) \subset \Sigma^+ \cup \{0\}$. Hence, the root basis Π_a is adapted to Π . \square

From now on, we fix a root basis $\Pi_a = \{\gamma_s, s \in \bar{I}\}$, for the grading root system Σ , which is adapted to the root basis $\Pi = \{\alpha_i, i \in I\}$ of Δ (see Theorem 3.10). As before, let $J := \{j \in I; \rho_a(\alpha_j) = 0\}$ and $I' := I \setminus J$. For $k \in I'$, we denote, as above, by I_k the connected component of $J \cup \{k\}$ containing k , and $J_k := J \cap I_k$.

Proposition 3.11.

1) *Let $s \in \bar{I}$, then there exists $k_s \in I'$ such that $\rho_a(\alpha_{k_s}) = \gamma_s$ and any preimage $\alpha \in \Delta$ of γ_s is equal to α_k modulo $\sum_{j \in J_k} \mathbb{Z}\alpha_j$ for some $k \in I'$ satisfying $\rho_a(\alpha_k) = \gamma_s$.*

2) Let $k \in I'$ such that $\rho_a(\alpha_k)$ is a real root of Σ . Then $\rho_a(\alpha_k) \in \Pi_a$ is a simple root.

Proof. This result was proved by J. Nervi for affine algebras (see [16], Prop.2.3.10 and the proof of Prop. 2.3.12). The arguments used there are available for general Kac-Moody algebras. \square

We introduce the following notations :

$$\begin{aligned} I'_{re} &:= \{i \in I' ; \rho_a(\alpha_i) \in \Pi_a\} ; \quad I'_{im} := I' \setminus I'_{re}, \\ I_{re} &= \bigcup_{k \in I'_{re}} I_k ; \quad J_{re} = I_{re} \cap J = \bigcup_{k \in I'_{re}} J_k ; \quad J^\circ = J \setminus J_{re} \\ \Gamma_s &:= \{i \in I' ; \rho_a(\alpha_i) = \gamma_s\}, \forall s \in \bar{I}. \end{aligned}$$

Note that J° is not connected to I_{re} .

Remark 3.12.

- 1) In view of Proposition 3.11, assertion 2), one has $\rho_a(\alpha_k) \in \Sigma_{im}^+, \forall k \in I'_{im}$.
- 2) $I = I_{re} \cup I'_{im} \cup J^\circ$ is a disjoint union.
- 3) If $I'_{im} = \emptyset$, then $I = I_{re} \cup J^\circ$. Since I is connected (and I_{re} is not connected to J°) we deduce that $J^\circ = \emptyset$, $I = I_{re}$ and $I'_{re} = I' = I \setminus J$.

Proposition 3.13.

- 1) Let $k \in I'_{re}$, then I_k is of finite type.
- 2) Let $s \in \bar{I}$. If $|\Gamma_s| \geq 2$ and $k \neq l \in \Gamma_s$, then $I_k \cup I_l$ is not connected: $\mathfrak{g}(I_k)$ and $\mathfrak{g}(I_l)$ commute and are orthogonal.
- 3) For all $k \in I'_{re}$, (I_k, J_k) is an irreducible C -admissible pair.
- 4) The derived subalgebra \mathfrak{m}' of the grading algebra \mathfrak{m} is contained in $\mathfrak{g}'(I_{re})$ (as defined in proposition 1.2).

Proof.

- 1) Suppose that there exists $k \in I'_{re}$ such that I_k is not of finite type; then there exists an imaginary root β_k whose support is the whole I_k . Hence, there exists a positive integer $m_k \in \mathbb{N}$ such that $\rho_a(\beta_k) = m_k \rho(\alpha_k)$ is an imaginary root of Σ . It follows that $\rho_a(\alpha_k)$ is an imaginary root and this contradicts the fact that $k \in I'_{re}$.
- 2) Let $s \in \bar{I}$ such that $|\Gamma_s| \geq 2$ and let $k \neq l \in \Gamma_s$. Since $V_{n\gamma_s} = \{0\}$ for all integer $n \geq 2$, the same argument used in 1) shows that $I_k \cup I_l$ is not connected, and I_k and I_l are its two connected components. In particular, $[\mathfrak{g}(I_k), \mathfrak{g}(I_l)] = \{0\}$ and $(\mathfrak{g}(I_k), \mathfrak{g}(I_l)) = \{0\}$.
- 3) Let $k \in I'_{re}$ and let $s \in \bar{I}$ such that $\rho_a(\alpha_k) = \gamma_s$. Let $(\bar{X}_s, \bar{H}_s = \gamma_s^\vee, \bar{Y}_s)$ be an \mathfrak{sl}_2 -triple in \mathfrak{m} corresponding to the simple root γ_s . Let V_{γ_s} be the weight space of \mathfrak{g} corresponding to γ_s . In view of Proposition 3.11, assertion 1), one has :

$$(3.1) \quad V_{\gamma_s} = \bigoplus_{l \in \Gamma_s} V_{\gamma_s} \cap \mathfrak{g}(I_l).$$

Hence, one can write :

$$(3.2) \quad \bar{X}_s = \sum_{l \in \Gamma_s} E_l ; \quad \bar{Y}_s = \sum_{l \in \Gamma_s} F_l ,$$

with $E_l \in V_{\gamma_s} \cap \mathfrak{g}(I_l)$ and $F_l \in V_{-\gamma_s} \cap \mathfrak{g}(I_l)$. It follows from assertion 1) that

$$(3.3) \quad \bar{H}_s = \gamma_s^\vee = [\bar{X}_s, \bar{Y}_s] = \sum_{l \in \Gamma_s} [E_l, F_l] = \sum_{l \in \Gamma_s} H_l,$$

where $H_l := [E_l, F_l] \in \mathfrak{h}_l$, $\forall l \in \Gamma_s$. Then one has, for $k \in \Gamma_s$,

$$2 = \langle \gamma_s, \gamma_s^\vee \rangle = \langle \alpha_k, \gamma_s^\vee \rangle = \sum_{l \in \Gamma_s} \langle \alpha_k, H_l \rangle = \langle \alpha_k, H_k \rangle,$$

and for $j \in J_k$,

$$0 = \langle \alpha_j, \gamma_s^\vee \rangle = \sum_{l \in \Gamma_s} \langle \alpha_j, H_l \rangle = \langle \alpha_j, H_k \rangle.$$

In particular, H_k is the unique semi-simple element of \mathfrak{h}_{I_k} satisfying :

$$(3.4) \quad \langle \alpha_i, H_k \rangle = 2\delta_{i,k}, \forall i \in I_k.$$

Hence, (E_k, H_k, F_k) is an \mathfrak{sl}_2 -triple in the simple Lie algebra $\mathfrak{g}(I_k)$ and since $V_{2\gamma_s} = \{0\}$, (I_k, J_k) is an irreducible C -admissible pair for all $k \in \Gamma_s$.

The assertion 4) follows from the relation (3.2). \square

Corollary 3.14. *The pair (I_{re}, J_{re}) is C -admissible. If $I'_{im} = \emptyset$, then $I_{re} = I$, $J_{re} = J$ and \mathfrak{g} is finitely Δ^J -graded, with grading subalgebra \mathfrak{g}^J .*

N.B. We have got part 2) of Theorem 2.

Proof. The first assertion is a consequence of Proposition 3.13. By remark 3.12, when $I'_{im} = \emptyset$, we have $I = I_{re}$; hence, by Theorem 2.14, \mathfrak{g} is finitely Δ^J -graded. \square

Definition 3.15. If $I'_{im} \neq \emptyset$, then (I, J) is called a generalized C -admissible pair. If $I'_{im} = J = \emptyset$, the Kac-Moody algebra \mathfrak{g} is said to be maximally finitely Σ -graded.

Corollary 3.16. *The grading subalgebra \mathfrak{m} of \mathfrak{g} is symmetrizable and the restriction to \mathfrak{m} of the invariant bilinear form of \mathfrak{g} is nondegenerate.*

Proof. Let $(\cdot, \cdot)_a$ be the restriction to \mathfrak{m} of the invariant bilinear form (\cdot, \cdot) of \mathfrak{g} . Recall from the proof of Proposition 3.13 that $\gamma_s^\vee = \sum_{k \in \Gamma_s} H_k$, $\forall s \in \bar{I}$. In particular $(\gamma_s^\vee, \gamma_s^\vee)_a = \sum_{k \in \Gamma_s} (H_k, H_k) > 0$. It follows that $(\cdot, \cdot)_a$ is a nondegenerate invariant bilinear form on \mathfrak{m} (see §1.4) and that \mathfrak{m} is symmetrizable. \square

Corollary 3.17. *Let \mathfrak{h}^J be the orthogonal of \mathfrak{h}_J in \mathfrak{h} . For $k \in I'_{im}$, write*

$$\rho_a(\alpha_k) = \sum_{s \in \bar{I}} n_{s,k} \gamma_s.$$

For $s \in \bar{I}$, choose l_s a representative element of Γ_s . Then $\mathfrak{a}/\mathfrak{c}_a$ can be viewed as the subspace of $\mathfrak{h}^J/\mathfrak{c}$ defined by the following relations :

$$\langle \alpha_k, h \rangle = \langle \alpha_{l_s}, h \rangle, \forall k \in \Gamma_s, \forall s \in \bar{I}$$

$$\langle \alpha_k, h \rangle = \sum_{s \in \bar{I}} n_{s,k} \langle \alpha_{l_s}, h \rangle, \forall k \in I'_{im}.$$

Proof. The subspace of $\mathfrak{h}^J/\mathfrak{c}$ defined by the above relations has dimension $|\bar{I}|$ and contains $\mathfrak{a}/\mathfrak{c}_a$ and hence it is equal to $\mathfrak{a}/\mathfrak{c}_a$. \square

Proposition 3.18. *Let $(\cdot, \cdot)_a$ be the restriction to \mathfrak{m} of the invariant bilinear form (\cdot, \cdot) of \mathfrak{g} .*

- 1) *Let $\mathfrak{a}' = \mathfrak{a} \cap \mathfrak{m}'$ and let \mathfrak{a}'' be a supplementary subspace of \mathfrak{a}' in \mathfrak{a} which is totally isotropic relatively to $(\cdot, \cdot)_a$. Then $\mathfrak{a}'' \cap \mathfrak{h}' = \{0\}$.*
- 2) *Let $A_{I_{re}}$ be the submatrix of A indexed by I_{re} . Then there exists a subspace $\mathfrak{h}_{I_{re}}$*

of \mathfrak{h} containing \mathfrak{a} such that $(\mathfrak{h}_{I_{re}}, \Pi_{I_{re}}, \Pi_{I_{re}}^\vee)$ is a realization of $A_{I_{re}}$. In particular, the Kac-Moody subalgebra $\mathfrak{g}(I_{re})$ associated to this realization (in 1.2) contains the grading subalgebra \mathfrak{m} .

3) The Kac-Moody algebra $\mathfrak{g}(I_{re})$ is finitely $\Delta(I_{re})^{J_{re}}$ -graded and its grading subalgebra is the subalgebra $\mathfrak{g}(I_{re})^{J_{re}}$ associated to the C -admissible pair (I_{re}, J_{re}) as in Proposition 2.11.

4) The Kac-Moody algebra $\mathfrak{g}(I_{re})^{J_{re}}$ contains \mathfrak{m} .

Proof.

1) Recall that the center \mathfrak{c}_a of \mathfrak{m} is contained in the center \mathfrak{c} of \mathfrak{g} . Since $\mathfrak{h}' = \mathfrak{c}^\perp$ and \mathfrak{c}_a is in duality with \mathfrak{a}'' relatively to $(\cdot, \cdot)_a$, we deduce that $\mathfrak{a}'' \cap \mathfrak{h}' = \{0\}$.

2) From the proofs of 3.16 and 3.13 we get $\gamma_s^\vee = \sum_{k \in \Gamma_s} H_k \in \sum_{k \in \Gamma_s} \mathfrak{h}_{I_k} = \mathfrak{h}'_{I_{re}}$. So $\mathfrak{c}_a \subset \mathfrak{a}' \subset \mathfrak{h}'_{I_{re}} \subset \mathfrak{h}'$. It follows that $(\mathfrak{h}'_{I_{re}} + \mathfrak{h}^{I_{re}})$ is contained in \mathfrak{c}_a^\perp the orthogonal subspace of \mathfrak{c}_a in \mathfrak{h} . Since $\mathfrak{a}'' \cap \mathfrak{c}_a^\perp = \{0\}$, one can choose a supplementary subspace $\mathfrak{h}''_{I_{re}}$ of $(\mathfrak{h}'_{I_{re}} + \mathfrak{h}^{I_{re}})$ containing \mathfrak{a}'' . Let $\mathfrak{h}_{I_{re}} = \mathfrak{h}'_{I_{re}} \oplus \mathfrak{h}''_{I_{re}}$, then, by Proposition 1.2, $(\mathfrak{h}_{I_{re}}, \Pi_{I_{re}}, \Pi_{I_{re}}^\vee)$ is a realization of $A_{I_{re}}$.

3) As in Corollary 3.14, assertion 3) is a simple consequence of Theorem 2.14.

4) The algebra \mathfrak{a} is in $\mathfrak{h}_{I_{re}} \cap \Pi_J^\perp = (\mathfrak{h}_{I_{re}})^{J_{re}}$. By the proof of Proposition 3.13, for $s \in \bar{I}$, \bar{X}_s and \bar{Y}_s are linear combinations of the elements in $\{E_k, F_k \mid k \in \Gamma_s\} \subset \mathfrak{g}(I_{re})^{J_{re}}$. Hence $\mathfrak{g}(I_{re})^{J_{re}}$ contains all generators of \mathfrak{m} . \square

Lemma 3.19. *Let \mathfrak{l} be a Kac-Moody subalgebra of \mathfrak{g} containing \mathfrak{m} . Then \mathfrak{l} is finitely Σ -graded. In particular, the Kac-Moody subalgebra $\mathfrak{g}(I_{re})$ or $\mathfrak{g}(I_{re})^{J_{re}}$ is finitely Σ -graded.*

N.B. Proposition 3.18 and Lemma 3.19 finish the proof of Theorem 2.

Proof. Recall that the Cartan subalgebra \mathfrak{a} of \mathfrak{m} is $\text{ad}_{\mathfrak{g}}$ -diagonalizable. Since \mathfrak{l} is $\text{ad}(\mathfrak{a})$ -invariant, one has $\mathfrak{l} = \sum_{\gamma \in \Sigma \cup \{0\}} V_\gamma \cap \mathfrak{l}$. By assumption $\{0\} \neq \mathfrak{m}_\gamma \subset V_\gamma \cap \mathfrak{l}$ for all $\gamma \in \Sigma$; hence, we deduce that \mathfrak{l} is finitely Σ -graded. \square

Proposition 3.20. *If \mathfrak{g} is of finite, affine or hyperbolic type, then $I'_{im} = \emptyset$ and (I, J) is a C -admissible pair.*

Proof. The result is trivial if \mathfrak{g} is of finite type. Suppose $I'_{im} \neq \emptyset$ for one of the other cases. If \mathfrak{g} is affine, then I_{re} is of finite type and by Lemma 3.18, \mathfrak{m} is contained in the finite dimensional semi-simple Lie algebra $\mathfrak{g}(I_{re})$. This contradicts the fact that \mathfrak{m} is, as \mathfrak{g} , of affine type (see Proposition 3.3). If \mathfrak{g} is hyperbolic, then it is Lorentzian and perfect (cf. section 1.1), and by Lemma 3.19, $\mathfrak{g}(I_{re})$ is a finitely Σ -graded subalgebra of \mathfrak{g} . As I_{re} is assumed to be a proper subset of I , $\mathfrak{g}(I_{re})$ is of finite or affine type. This contradicts Proposition 3.3, since \mathfrak{m} should be Lorentzian (cf. 3.3). Hence, $I'_{im} = \emptyset$ in the two last cases. \square

Proposition 3.21. *If \mathfrak{g} is of hyperbolic type, then the grading subalgebra \mathfrak{m} is also of hyperbolic type.*

Proof. Recall that in this case, $I_{re} = I$ (see Proposition 3.20 and Corollary 3.14). Let \bar{I}^1 be a proper subset of \bar{I} and suppose that \bar{I}^1 is connected. Let $I^1 = \bigcup_{s \in \bar{I}^1} (\bigcup_{k \in \Gamma_s} I_k)$. Then, I^1 is a proper subset of I . We may assume that the subalgebra $\mathfrak{m}(\bar{I}^1)$ of \mathfrak{m} is contained in $\mathfrak{g}(I^1)$. Let $\Sigma^1 := \Sigma(\bar{I}^1)$ be the root system of $\mathfrak{m}(\bar{I}^1)$. Then, it is not difficult to check that $\mathfrak{g}(I^1)$ is Σ^1 -graded. Since $\mathfrak{g}(I^1)$ is of

finite or affine type, we deduce, by Proposition 3.3, that $\mathfrak{m}(\bar{I}^1)$ is of finite or affine type. Hence, \mathfrak{m} is hyperbolic. \square

Proposition 3.22. *If $I'_{im} = \emptyset$, then $\mathfrak{g}(I_{re}) = \mathfrak{g}$ and the C -admissible subalgebra \mathfrak{g}^J is maximally finitely Σ -graded, with grading subalgebra \mathfrak{m} .*

Proof. This result is due to J. Nervi ([16]; Thm 2.5.10) for the affine case; it follows from the facts that $V_0 \cap \mathfrak{g}^J = \mathfrak{h}^J$ and $\mathfrak{m} \subset \mathfrak{g}^J$ (see Prop. 3.18). \square

Corollary 3.23. *If \mathfrak{g} is of finite, affine or hyperbolic type, the problem of classification of finite gradations of \mathfrak{g} comes down first to classify the C -admissible pairs (I, J) of \mathfrak{g} and then the maximal finite gradations of the corresponding admissible algebra \mathfrak{g}^J .*

Proof. This follows from Proposition 3.20, Proposition 3.22 and Lemma 1.5. \square

4. MAXIMAL GRADATIONS

We assume now that \mathfrak{g} is maximally finitely Σ -graded. We keep the notations in section 3 but we have $J = I'_{im} = \emptyset$. So \bar{I} is a quotient of I , with quotient map ρ defined by $\rho_a(\alpha_k) = \gamma_{\rho(k)}$. For $s \in \bar{I}$, $\Gamma_s = \rho^{-1}(\{s\})$.

Proposition 4.1.

- 1) If $k \neq l \in I$ and $\rho(k) = \rho(l)$, then there is no link between k and l in the Dynkin diagram of A : $\alpha_k(\alpha_l^\vee) = \alpha_l(\alpha_k^\vee) = 0$ and $(\alpha_k, \alpha_l) = 0$.
- 2) $\mathfrak{a} \subset \{h \in \mathfrak{h} \mid \alpha_k(h) = \alpha_l(h) \text{ whenever } \rho(k) = \rho(l)\}$.
- 3) For good choices of the simple coroots and Chevalley generators $(\alpha_k^\vee, e_k, f_k)_{k \in I}$ in \mathfrak{g} and $(\gamma_s^\vee, \bar{X}_s, \bar{Y}_s)_{s \in \bar{I}}$ in \mathfrak{m} , we have $\gamma_s^\vee = \sum_{k \in \Gamma_s} \alpha_k^\vee$, $\bar{X}_s = \sum_{k \in \Gamma_s} e_k$ and $\bar{Y}_s = \sum_{k \in \Gamma_s} f_k$.
- 4) In particular, for $s, t \in \bar{I}$, we have $\gamma_s(\gamma_t^\vee) = \sum_{k \in \Gamma_t} \alpha_i(\alpha_k^\vee)$ for any $i \in \Gamma_s$.

Proof. Assertions 1) and 2) are proved in 3.13 and 3.17. For $i \in \Gamma_s$, $\gamma_s = \rho_a(\alpha_i)$ is the restriction of α_i to \mathfrak{a} ; so 4) is a consequence of 3).

For 3) recall the proof of Proposition 3.13. The \mathfrak{sl}_2 -triple $(\bar{X}_s, \gamma_s^\vee, \bar{Y}_s)$ may be written $\gamma_s^\vee = \sum_{k \in \Gamma_s} H_k$, $\bar{X}_s = \sum_{k \in \Gamma_s} E_k$ and $\bar{Y}_s = \sum_{k \in \Gamma_s} F_k$ where (E_k, H_k, F_k) is an \mathfrak{sl}_2 -triple in $\mathfrak{g}(I_k)$, with $\alpha_k(H_k) = 2$. But now $J = I'_{im} = \emptyset$, so $I_k = \{k\}$ and $\mathfrak{g}(I_k) = \mathbb{C}e_k \oplus \mathbb{C}\alpha_k^\vee \oplus \mathbb{C}f_k$, hence the result. \square

So the grading subalgebra \mathfrak{m} may be entirely described by the quotient map ρ .

We look now to the reciprocal construction.

So \mathfrak{g} is an indecomposable and symmetrizable Kac-Moody algebra associated to a generalized Cartan matrix $A = (a_{i,j})_{i,j \in I}$. We consider a quotient \bar{I} of I with quotient map $\rho : I \rightarrow \bar{I}$ and fibers $\Gamma_s = \rho^{-1}(\{s\})$ for $s \in \bar{I}$. We suppose that ρ is an admissible quotient i.e. that it satisfies the following two conditions:

(MG1) If $k \neq l \in I$ and $\rho(k) = \rho(l)$, then $a_{k,l} = \alpha_l(\alpha_k^\vee) = 0$.

(MG2) If $s \neq t \in \bar{I}$, then $\bar{a}_{s,t} := \sum_{i \in \Gamma_s} a_{i,j} = \sum_{i \in \Gamma_s} \alpha_j(\alpha_i^\vee)$ is independent of the choice of $j \in \Gamma_t$.

Proposition 4.2. *The matrix $\bar{A} = (\bar{a}_{s,t})_{s,t \in \bar{I}}$ is an indecomposable generalized Cartan matrix.*

Proof. Let $s \neq t \in \bar{I}$ and let $j \in \Gamma_t$. By (MG1) one has $\bar{a}_{t,t} = \sum_{i \in \Gamma_t} a_{i,j} = a_{j,j} = 2$, and by (MG2) $\bar{a}_{s,t} := \sum_{i \in \Gamma_s} a_{i,j} \in \mathbb{Z}^-$ ($\forall j \in \Gamma_t$). Moreover, $\bar{a}_{s,t} = 0$ if and only if $a_{i,j} = 0 (= a_{j,i})$, $\forall (i,j) \in \Gamma_s \times \Gamma_t$. It follows that $\bar{a}_{s,t} = 0$ if and only if $\bar{a}_{t,s} = 0$, and \bar{A} is a generalized Cartan matrix. Since A is indecomposable, \bar{A} is also indecomposable. \square

Let $\mathfrak{h}^\Gamma = \{h \in \mathfrak{h} \mid \alpha_k(h) = \alpha_l(h) \text{ whenever } \rho(k) = \rho(l)\}$, $\gamma_s^\vee = \sum_{k \in \Gamma_s} \alpha_k^\vee$ and $\mathfrak{a}' = \bigoplus_{s \in \bar{I}} \mathbb{C} \gamma_s^\vee \subset \mathfrak{h}^\Gamma$. We may choose a subspace \mathfrak{a}'' in \mathfrak{h}^Γ such that $\mathfrak{a}'' \cap \mathfrak{a}' = \{0\}$, the restrictions $\bar{\alpha}_i =: \gamma_{\rho(i)}$ to $\mathfrak{a} = \mathfrak{a}' \oplus \mathfrak{a}''$ of the simple roots α_i (corresponding to different $\rho(i) \in \bar{I}$) are linearly independent and \mathfrak{a}'' is minimal for these two properties.

Proposition 4.3. $(\mathfrak{a}, \{\gamma_s \mid s \in \bar{I}\}, \{\gamma_s^\vee \mid s \in \bar{I}\})$ is a realization of \bar{A} .

Proof. Let ℓ be the rank of \bar{A} . Note that \mathfrak{a} contains $\mathfrak{a}' = \bigoplus_{s \in \bar{I}} \mathbb{C} \gamma_s^\vee$; the family $(\gamma_s)_{s \in \bar{I}}$ is free in the dual space \mathfrak{a}^* of \mathfrak{a} and satisfies $\langle \gamma_t, \gamma_s^\vee \rangle = \bar{a}_{s,t}$, $\forall s, t \in \bar{I}$. It follows that $\dim(\mathfrak{a}) \geq 2|\bar{I}| - \ell$ (see [[10]; Prop. 14.1] or [[11]; Exer. 1.3]). As \mathfrak{a} is minimal, we have $\dim(\mathfrak{a}) = 2|\bar{I}| - \ell$ (see [[10]; Prop. 14.2] for minimal realization). Hence $(\mathfrak{a}, \{\gamma_s \mid s \in \bar{I}\}, \{\gamma_s^\vee \mid s \in \bar{I}\})$ is a (minimal) realization of \bar{A} . \square

We note $\Delta^\rho = \Sigma \subset \bigoplus_{s \in \bar{I}} \mathbb{Z} \gamma_s$ the root system associated to this realization.

We define now $\bar{X}_s = \sum_{k \in \Gamma_s} e_k$ and $\bar{Y}_s = \sum_{k \in \Gamma_s} f_k$. Let $\mathfrak{m} = \mathfrak{g}^\rho$ be the Lie subalgebra of \mathfrak{g} generated by \mathfrak{a} and the elements \bar{X}_s, \bar{Y}_s for $s \in \bar{I}$.

Proposition 4.4. The Lie subalgebra $\mathfrak{m} = \mathfrak{g}^\rho$ is the Kac-Moody algebra associated to the realization $(\mathfrak{a}, \{\gamma_s \mid s \in \bar{I}\}, \{\gamma_s^\vee \mid s \in \bar{I}\})$ of \bar{A} . Moreover, \mathfrak{g} is an integrable \mathfrak{g}^ρ -module with finite multiplicities.

Proof. Clearly, the following relations hold in the Lie subalgebra \mathfrak{g}^ρ :

$$\begin{aligned} [\mathfrak{a}, \mathfrak{a}] &= 0, & [\bar{X}_s, \bar{Y}_t] &= \delta_{s,t} \gamma_s^\vee & (s, t \in \bar{I}); \\ [a, \bar{X}_s] &= \langle \gamma_s, a \rangle \bar{X}_s, & [a, \bar{Y}_s] &= -\langle \gamma_s, a \rangle \bar{Y}_s & (a \in \mathfrak{a}, s \in \bar{I}). \end{aligned}$$

For the Serre's relations, one has :

$$1 - \bar{a}_{s,t} \geq 1 - a_{i,j}, \quad \forall (i,j) \in \Gamma_s \times \Gamma_t.$$

In particular, one can see, by induction on $|\Gamma_s|$, that :

$$(\text{ad} \bar{X}_s)^{1-\bar{a}_{s,t}}(e_j) = \left(\sum_{i \in \Gamma_s} \text{ad} e_i \right)^{1-\bar{a}_{s,t}}(e_j) = 0, \quad \forall j \in \Gamma_t.$$

Hence

$$(\text{ad} \bar{X}_s)^{1-\bar{a}_{s,t}}(\bar{X}_t) = 0, \quad \forall s, t \in \bar{I},$$

and in the same way we obtain that :

$$(\text{ad} \bar{Y}_s)^{1-\bar{a}_{s,t}}(\bar{Y}_t) = 0, \quad \forall s, t \in \bar{I}.$$

It follows that \mathfrak{g}^ρ is a quotient of the Kac-Moody algebra $\mathfrak{g}(\bar{A})$ associated to \bar{A} and $(\mathfrak{a}, \{\gamma_s \mid s \in \bar{I}\}, \{\gamma_s^\vee \mid s \in \bar{I}\})$ in which the Cartan subalgebra \mathfrak{a} of $\mathfrak{g}(\bar{A})$ is embedded. By [11, 1.7] \mathfrak{g}^ρ is equal to $\mathfrak{g}(\bar{A})$.

It's clear that \mathfrak{g} is an integrable \mathfrak{g}^ρ -module with finite dimensional weight spaces

relative to the adjoint action of \mathfrak{a} , since for $\alpha = \sum_{i \in I} n_i \alpha_i \in \Delta^+$, its restriction to \mathfrak{a} , is given by

$$(4.1) \quad \rho_a(\alpha) = \sum_{s \in \bar{I}} \left(\sum_{i \in \Gamma_s} n_i \right) \gamma_s$$

□

Proposition 4.5. *The Kac-Moody algebra \mathfrak{g} is maximally finitely Δ^ρ -graded with grading subalgebra \mathfrak{g}^ρ .*

Proof. As in Theorem 2.14, we will see that $\rho_a(\Delta^+) \subset Q_+^\Gamma := \bigoplus_{s \in \bar{I}} \mathbb{Z}^+ \gamma_s$ satisfies, as

$\Sigma^+ = \Delta_+^\rho$, the following conditions :

- (i) $\gamma_s \in \rho_a(\Delta^+) \subset Q_+^\Gamma$, $2\gamma_s \notin \rho_a(\Delta^+)$, $\forall s \in \bar{I}$.
- (ii) if $\gamma \in \rho_a(\Delta^+)$, $\gamma \neq \gamma_s$, then the set $\{\gamma + k\gamma_s; k \in \mathbb{Z}\} \cap \rho_a(\Delta^+)$ is a string $\{\gamma - p\gamma_s, \dots, \gamma + q\gamma_s\}$, where $p, q \in \mathbb{Z}^+$ and $p - q = \langle \gamma, \gamma_s^\vee \rangle$;
- (iii) if $\gamma \in \rho_a(\Delta^+)$, then $\text{supp}(\gamma)$ is connected.

Clearly $\{\gamma_s \mid s \in \bar{I}\} \subset \rho_a(\Delta_+) \subset Q_+^\Gamma$. For $\alpha \in \Delta$ and $s \in \bar{I}$, the condition $\rho_a(\alpha) \in \mathbb{N}\gamma_s$ implies $\alpha \in \Delta(\Gamma_s)^+ = \{\alpha_i; i \in \Gamma_s\}$ [see (4.1)]. It follows that $2\gamma_s \notin \rho_a(\Delta_+)$ and (i) is satisfied. By Proposition 4.4, \mathfrak{g} is an integrable \mathfrak{g}^ρ -module with finite multiplicities. Hence, the propriety (ii) follows from [[11]; prop.3.6]. Let $\alpha \in \Delta_+$ and let $s, t \in \text{supp}(\rho_a(\alpha))$. By (4.1) there exists $(k, l) \in \Gamma_s \times \Gamma_t$ such that $k, l \in \text{supp}(\alpha)$, which is connected. Hence there exist $i_0 = k, i_1, \dots, i_{n+1} = l$ such that $\alpha_{i_j} \in \text{supp}(\alpha)$, $j = 0, 1, \dots, n+1$, and for $j = 0, 1, \dots, n$, i_j and i_{j+1} are linked relative to the generalized Cartan matrix A . In particular, $\rho(i_j) \neq \rho(i_{j+1}) \in \text{supp}(\rho_a(\alpha))$ and they are linked relative to the generalized Cartan matrix \bar{A} , $j = 0, 1, \dots, n$, with $\rho(i_0) = s$ and $\rho(i_{n+1}) = t$. Hence the connectedness of $\text{supp}(\rho_a(\alpha))$ relative to \bar{A} . It follows that $\rho_a(\Delta^+) = \Delta_+^\rho$ and hence $\rho_a(\Delta) = \Delta^\rho$ (see [11], Ex. 5.4). In particular, \mathfrak{g} is finitely Δ^ρ -graded with $J = \emptyset = I'_{im}$.

□

Corollary 4.6. *The restriction to $\mathfrak{m} = \mathfrak{g}^\rho$ of the invariant bilinear form (\cdot, \cdot) of \mathfrak{g} is nondegenerate. In particular, the generalized Cartan matrix \bar{A} is symmetrizable of the same type as A .*

Proof. The first part of the corollary follows from Proposition 4.5 and Corollary 3.16. The second part follows from Proposition 3.3. □

Remark 4.7. The map ρ coincides with the map (also denoted ρ) defined at the beginning of this section using the maximal gradation of Proposition 4.5. Conversely Proposition 4.1 tells that, for a general maximal gradation, ρ is admissible and $\mathfrak{m} = \mathfrak{g}^\rho$ for good choices of the Chevalley generators. So we get a good correspondence between maximal gradations and admissible quotient maps.

5. AN EXAMPLE

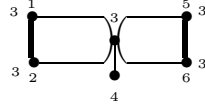
The following example shows that generalized C -admissible pairs do exist. It shows in particular that, for a generalized C -admissible pair (I, J) , J° may be nonempty and I_{re} may be non connected. Moreover, the Kac-Moody algebra may be not graded by the root system of $\mathfrak{g}(I_{re})$.

Gradations revealing generalized C -admissible pairs will be studied in a forthcoming paper.

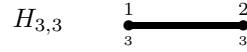
Example 5.1. Consider the Kac Moody algebra \mathfrak{g} corresponding to the indecomposable and symmetric generalized Cartan matrix A :

$$A = \begin{pmatrix} 2 & -3 & -1 & 0 & 0 & 0 \\ -3 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & -3 \\ 0 & 0 & -1 & 0 & -3 & 2 \end{pmatrix}$$

with the corresponding Dynkin diagram :



Note that $\det(A) = 275$ and the symmetric submatrix of A indexed by $\{1, 2, 4, 5, 6\}$ has signature $(+, +, +, -, -)$. Since $\det(A) > 0$, the matrix A should have signature $(+, +, +, -, -)$. Let Σ be the root system associated to the strictly hyperbolic generalized Cartan matrix $\begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$, the corresponding Dynkin diagram is the following :

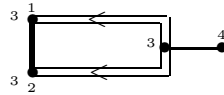


We will see that \mathfrak{g} is finitely Σ -graded and describe the corresponding generalized C -admissible pair.

1) Let τ be the involutive diagram automorphism of \mathfrak{g} such that $\tau(1) = 5$, $\tau(2) = 6$ and τ fixes the other vertices. Let σ'_n be the normal semi-involution of \mathfrak{g} corresponding to the split real form of \mathfrak{g} . Consider the quasi-split real form $\mathfrak{g}_{\mathbb{R}}^1$ associated to the semi-involution $\tau\sigma'_n$ (see [2] or [6]). Then $\mathfrak{t}_{\mathbb{R}} := \mathfrak{h}_{\mathbb{R}}^{\tau}$ is a maximal split toral subalgebra of $\mathfrak{g}_{\mathbb{R}}^1$. The corresponding restricted root system $\Delta' := \Delta(\mathfrak{g}_{\mathbb{R}}, \mathfrak{t}_{\mathbb{R}})$ is reduced and the corresponding generalized Cartan matrix A' is given by :

$$A' = \begin{pmatrix} 2 & -3 & -2 & 0 \\ -3 & 2 & -2 & 0 \\ -1 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

with the corresponding Dynkin diagram :



Following N. Bardy [[4], §9], there exists a split real Kac-Moody subalgebra $\mathfrak{m}_{\mathbb{R}}^1$ of $\mathfrak{g}_{\mathbb{R}}^1$ containing $\mathfrak{t}_{\mathbb{R}}$ such that $\Delta' = \Delta(\mathfrak{m}_{\mathbb{R}}^1, \mathfrak{t}_{\mathbb{R}})$. It follows that \mathfrak{g} is finitely Δ' -graded.

2) Let $\mathfrak{m}^1 := \mathfrak{m}_{\mathbb{R}}^1 \otimes \mathbb{C}$ and $\mathfrak{t} := \mathfrak{t}_{\mathbb{R}} \otimes \mathbb{C}$. Denote by $\alpha'_i := \alpha_i/\mathfrak{t}$, $i = 1, 2, 3, 4$. Put $\alpha'_1 = \alpha_1 + \alpha_5$, $\alpha'_2 = \alpha_2 + \alpha_6$, $\alpha'_3 = \alpha_3$ and $\alpha'_4 = \alpha_4$. Let $I^1 := \{1, 2, 3, 4\}$, then $(\mathfrak{t}, \Pi' = \{\alpha'_i, i \in I^1\}, \Pi'^{\vee} = \{\alpha'^{\vee}_i, i \in I^1\})$ is a realization of A' which is symmetrizable and Lorentzian.

Let \mathfrak{m} be the Kac-Moody subalgebra of \mathfrak{m}^1 corresponding to the submatrix \bar{A} of A' indexed by $\{1, 2\}$. Thus $\bar{A} = \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$ is strictly hyperbolic. Let $\mathfrak{a} :=$

$\mathbb{C}\alpha'_1 \oplus \mathbb{C}\alpha'_2$ be the standard Cartan subalgebra of \mathfrak{m} and let $\Sigma = \Delta(\mathfrak{m}, \mathfrak{a})$. For $\alpha' \in \mathfrak{t}^*$, denote by $\rho_1(\alpha')$ the restriction of α' to \mathfrak{a} . Put $\gamma_s = \rho_1(\alpha'_s)$, $\gamma_s = \alpha'_s$, $s = 1, 2$. Then $\Pi_a = \{\gamma_1, \gamma_2\}$ is the standard root basis of Σ . One can see easily that $\rho_1(\alpha'_4) = 0$ and $\rho_1(\alpha'_3) = 2(\gamma_1 + \gamma_2)$ is a strictly positive imaginary root of Σ . Now we will show that \mathfrak{m}^1 is finitely Σ -graded.

Let $(\cdot, \cdot)_1$ be the normalized invariant bilinear form on \mathfrak{m}^1 such that short real roots have length 1 and long real roots have square length 2. Then there exists a positive rational q such that the restriction of $(\cdot, \cdot)_1$ to \mathfrak{t} has the matrix B_1 in the basis Π^\sim , where :

$$B_1 = q \begin{pmatrix} 2 & -3 & -1 & 0 \\ -3 & 2 & -1 & 0 \\ -1 & -1 & 1 & -1/2 \\ 0 & 0 & -1/2 & 1 \end{pmatrix}$$

By duality, the restriction of $(\cdot, \cdot)_1$ to \mathfrak{t} induces a nondegenerate symmetric bilinear form on \mathfrak{t}^* (see [11]; §2.1) such that its matrix B'_1 in the basis Π' , is the following :

$$B'_1 = q^{-1} \begin{pmatrix} 2 & -3 & -2 & 0 \\ -3 & 2 & -2 & 0 \\ -2 & -2 & 4 & -2 \\ 0 & 0 & -2 & 4 \end{pmatrix}$$

Hence, q equals 2.

Note that for $\alpha' = \sum_{i=1}^4 n_i \alpha'_i \in \Delta'^+$, we have that

$$(5.1) \quad (\alpha', \alpha')_1 = n_1^2 + n_2^2 + 2n_3^2 + 2n_4^2 - 3n_1n_2 - 2n_1n_3 - 2n_2n_3 - 2n_3n_4.$$

We will show that $\rho_1(\Delta'^+) = \Sigma^+ \cup \{0\}$. Note that Σ can be identified with $\Delta' \cap (\mathbb{Z}\alpha'_1 + \mathbb{Z}\alpha'_2)$; hence ρ_1 is injective on Σ and $\Sigma^+ \subset \rho_1(\Delta'^+)$.

Let $(\cdot, \cdot)_a$ be the normalized invariant bilinear form on \mathfrak{m} such that all real roots have length 2. Then the restriction of $(\cdot, \cdot)_a$ to \mathfrak{a} has the matrix B_a in the basis $\Pi_a = \{\gamma_1, \gamma_2\}$, where :

$$B_a = \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$$

Since \bar{A} is symmetric, the nondegenerate symmetric bilinear form, on \mathfrak{a}^* , induced by the restriction of $(\cdot, \cdot)_a$ to \mathfrak{a} , has the same matrix B_a in the basis Π_a . In particular, we have that :

$$(\rho_1(\alpha'), \rho_1(\alpha'))_a = 2[(n_1 + 2n_3)^2 + (n_2 + 2n_3)^2 - 3(n_1 + 2n_3)(n_2 + 2n_3)],$$

since $\rho_1(\alpha') = (n_1 + 2n_3)\gamma_1 + (n_2 + 2n_3)\gamma_2$.

Using (5.1), it is not difficult to check that

$$(5.2) \quad (\rho_1(\alpha'), \rho_1(\alpha'))_a = 2[(\alpha', \alpha')_1 - (n_3 - n_4)^2 - 5n_3^2 - n_4^2]$$

Suppose $n_3 = 0$, then, since $\text{supp}(\alpha')$ is connected, we have that $\alpha' = n_1\alpha'_1 + n_2\alpha'_2$ or $\alpha' = \alpha'_4$. Hence $\rho_1(\alpha') = n_1\gamma_1 + n_2\gamma_2 \in \Sigma$ or $\rho_1(\alpha') = 0$.

Suppose $n_3 \neq 0$, then, since $(\alpha', \alpha')_1 \leq 2$, one can see, using (5.2), that

$$(\rho_1(\alpha'), \rho_1(\alpha'))_a < 0.$$

As Σ is hyperbolic and $\rho_1(\alpha') \in \mathbb{N}\gamma_1 + \mathbb{N}\gamma_2$, we deduce that $\rho_1(\alpha')$ is a positive imaginary root of Σ (see [11]; Prop. 5.10). It follows that $\rho_1(\Delta'^+) = \Sigma^+ \cup \{0\}$.

To see that \mathfrak{m}^1 is finitely Σ -graded, it suffices to prove that, for $\gamma = m_1\gamma_1 + m_2\gamma_2 \in \Sigma^+ \cup \{0\}$, the set $\{\alpha' \in \Delta'^+, \rho_1(\alpha') = \gamma\}$ is finite. Note that if $\alpha' = \sum_{i=1}^4 n_i \alpha'_i \in \Delta'^+$ satisfying $\rho_1(\alpha') = \gamma$, then $n_i + 2n_3 = m_i$, $i = 1, 2$. In particular, there are

only finitely many possibilities for n_i , $i = 1, 2, 3$. The same argument as the one used in the proof of Proposition 2.13 shows also that there are only finitely many possibilities for n_4 .

3) Recall that $\mathfrak{m} \subset \mathfrak{m}^1 \subset \mathfrak{g}$. The fact that \mathfrak{g} is finitely Δ' -graded with grading subalgebra \mathfrak{m}^1 and \mathfrak{m}^1 is finitely Σ -graded implies that \mathfrak{g} is finitely Σ -graded (cf. lemma 1.5). Let $I = \{1, 2, 3, 4, 5, 6\}$, then the root basis Π_a of Σ is adapted to the root basis Π of Δ and we have $I_{re} = \{1, 2, 5, 6\}$ (not connected), $\Gamma_1 = \{1, 5\}$, $\Gamma_2 = \{2, 6\}$, $J = \{4\}$, $J_{re} = \emptyset$, $I'_{im} = \{3\}$ and $J^\circ = J = \{4\}$. Note that, for this example, $\mathfrak{g}(I_{re})$, which is Σ -graded, is isomorphic to $\mathfrak{m} \times \mathfrak{m}$. This gradation corresponds to that of the pseudo-complex real form of $\mathfrak{m} \times \mathfrak{m}$ (i.e. the complex Kac-Moody algebra \mathfrak{m} viewed as real Lie algebra) by its restricted reduced root system. Since the pair $(I_3, J_3) = (\{3, 4\}, \{4\})$ is not admissible, it is not possible to bring back J° to the empty set i.e. to build a Kac-Moody algebra \mathfrak{g}^J grading finitely \mathfrak{g} and maximally finitely Σ -graded.

REFERENCES

- [1] B. Allison, G. Benkart and Y. Gao; Central extensions of Lie algebras graded by finite root systems, *Math. Ann.* **316** (2000), 499-527.
- [2] V. Back-Valente, N. Bardy-Panse, H. Ben Messaoud and G. Rousseau; Formes presque déployées d'algèbres de Kac-Moody, Classification et racines relatives. *J. of Algebra* 171 (1995) 43-96.
- [3] N. Bardy-Panse; Systèmes de racine infinis. Mémoire de la S.M.F 65 (1996).
- [4] N. Bardy; Sous-algèbres birégulières d'une algèbre de Kac-Moody-Borcherds. *Nagoya Math. J.* Vol 156 (1999) 1-83.
- [5] G. Benkart and E. Zelmanov; Lie algebras graded by finite root systems and intersection matrix algebras, *Invent. Math.* **126** (1996), 1-45.
- [6] H. Ben Messaoud; Almost split real forms for hyperbolic Kac-Moody Lie algebras. *J. Phys A. Math. Gen* 39 (2006) 13659-13690.
- [7] S. Berman and R. Moody; Lie algebras graded by finite root systems, *Invent. Math.* **108** (1992), 323-347.
- [8] R. E. Borcherds; Generalized Kac-Moody algebras. *J. of Algebra* 115 (1988) 501-512.
- [9] N. Bourbaki; Groupes et algèbres de Lie, Chap 4, 5 et 6, Paris.
- [10] R. Carter; Lie algebras of finite and affine type. Cambridge University Press (2005).
- [11] V.G. Kac; Infinite dimensional Lie algebras. Third edition, Cambridge University Press (1990).
- [12] V.G. Kac and S.P. Wang; On automorphisms of Kac-Moody algebras and groups. *Advances in Math.* 92 (1992) 129-195.
- [13] R.V. Moody; Root systems of hyperbolic type. *Advances in Mathematics* 33, 144-160 (1979).
- [14] E. Neher; Lie algebras graded by J -graded root systems and Jordan pairs covered by grids, *Amer. J. Math.* **118** (1996), 439-491.
- [15] J. Nervi; Algèbres de Lie simples graduées par un système de racines et sous-algèbres C -admissibles. *J. of Algebra* 223 (2000) 307-343.
- [16] J. Nervi; Affine Kac-Moody algebras graded by affine root systems. *J. of Algebra* 253 (2002) 50-99.
- [17] D.H. Peterson and V.G. Kac; Infinite flag varieties and conjugacy theorems. *Proc. Natl. Acad. Sc. USA* 80 (1983) 1778-1782.
- [18] G. Rousseau; Groupes de Kac-Moody déployés sur un corps local, II Mesures ordonnées. ArXiv 1009.0135v2.
- [19] H. Rubenthaler; Construction de certaines sous-algèbres remarquables dans les algèbres de Lie semi-simples. *J. of Algebra* 81 (1983) 268-278.
- [20] J. Tits; Uniqueness and presentation of Kac-Moody groups over fields. *J. of Algebra* 105 (1987) 542-573.

DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ DE MONASTIR, FACULTÉ DES SCIENCES, 5019
MONASTIR. TUNISIE.

INSTITUT ELIE CARTAN, UNIVERSITÉ DE LORRAINE, BP 70239, 54506 VANDOEUVRE LÈS-NANCY
CEDEX, FRANCE.

E-mail address: `hechmi.benmessaoud@fsm.rnu.tn`; `Guy.Rousseau@univ-lorraine.fr`